Introduction to DC Motors and Rotary Encoders[1]

Anatomy of a DC Servo Motor

A conceptual diagram of a standard DC brushed motor is shown in Fig. 1. We will go through each major part of the motor, describing its function in as few words as we can.

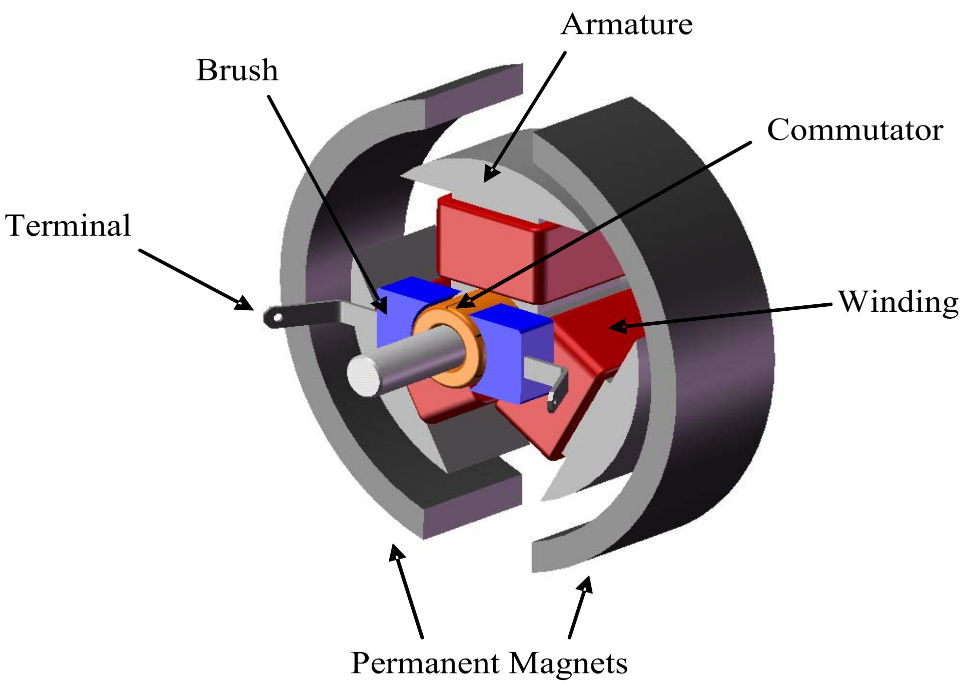

Figure 1: A permanent magnet brushed DC motor.

The Stator. The stator, the stationary outer part of a motor, is made up of an iron or steel housing filled with permanent magnets. The function of the stator is to create a magnetic field inside the motor.

The Rotor. The rotating part of the motor is called the rotor. The rotor is made up of an iron or steel armature that has several coils, called windings, spaced at even intervals along the rotor. As you may remember from freshman physics, a coil placed in a magnetic field will experience a torque proportional to the current flowing through it. The torque exerted by the motor is equal to the sum of the torques exerted on each of the windings by the magnetic field.

The Brushes. The brushes are a set of spring-loaded contacts that supply current to the rotor windings. They contact the rotor on a drum of electrical contacts called the commutator. The brushes and the commutator have the vital job of determining which windings receive current based on their orientation in the motor's magnetic field. Windings passing through the orientation in which they produce maximum torque are connected to the brushes, and the direction of the current is timed so that coils on the opposite sides of the rotor produce torques in the same direction. Inside a typical DC brushed motor, multiple windings are connected to the brushes at any given time.

1 This document is extracted from "2.12: Introduction to Robotics Wheeled Vehicle Lab Manual" by Lael Odhner, *et al*.

The Terminals. As a consumer of motors, the terminals are your point of interface with a motor. All incoming electrical power to the motor is provided through one set of terminals, which represent several of the motor windings connected in parallel. Because of this, the electrical properties of the motor are often prefaced with the word terminal (e.g. terminal resistance, terminal inductance). In all but the most exotic motors, the consumer of motors needs only these lumped parallel electrical properties to design around a motor.

The performance properties of a DC brushed motor are defined by a small number of electrical and mechanical parameters, shown in Table 1 below. In the pre-lab exercises for this week, we will ask you to collect these constants for the motors we will use on our wheeled vehicle, and to comment briefly on the performance capabilities of these motors.

Table 1: A List of Motor Parameters

Parameter	Symbol	SI Units	What it affects
Torque Constant	K_t	V·sec/rad or N·m/A	Torque-current and speed-voltage relationship
Terminal Resistance	R_m	Ω	Stall current and resistive losses
Rotor Inertia	I_m	kg·m^2	Maximum angular acceleration
Terminal Inductance	L_m	H	High-frequency electrical response
Maximum Continuous-Duty Current	i_{max}	A	Maximum torque produced
Maximum Continuous-Duty Voltage	V_{max}	V	Maximum speed produced

The Torque Constant, K_t

The most important parameter defining the performance of a motor is the torque constant, K_t. It is also sometimes referred to as the speed constant or the winding constant. The torque constant is fundamentally determined by the magnetic field strength and the properties of the motor windings: the number of turns of wire on each winding, the size of the winding and the number of windings connected to the brushes at any time by the commutator. For all practical purposes, though, it is just an empirical constant given by the manufacturer that relates the current input i to the torque output τ,

$$\tau = K_t \cdot i \tag{1}$$

The same model of motor will usually have many available windings, which yield different torque constants. For example, the Maxon F 2140 motors used on the wheeled vehicle have several different windings, of which we will be using the 937 winding[2]. Usually, the motor can be chosen by determining the desired continuous torque output of the motor, and comparing this to the maximum current of the motor multiplied by the torque constant:

2 The winding number is often appended onto the model number of the motor. In the case of the Maxon motors used in this lab, the part number on the side of the motor is F2140-937. This numbering convention varies by manufacturer.

$$\tau_{max} = K_t \cdot i_{max} \tag{2}$$

You will also notice from the motor data sheet for this lab that the maximum operating speed of a motor is inversely related to the torque constant. In fact, the torque constant defines the speed-voltage relationship just as directly as it does the torque-current relationship. This can be seen if we assume for a moment an ideal motor whose input electrical power written in terms of the voltage V and the current i is equal to the output mechanical power written in terms of the torque τ and the speed ω,

$$\tau \cdot \omega = V \cdot i \tag{3}$$

If this power conservation relationship is rewritten in terms of the torque constant using (1), we find that we can write a relationship between V and ω that does not depend on τ or i,

$$\begin{gathered} \tau \cdot \omega = (K_t \cdot i) \cdot \omega = V \cdot i \\ \omega = \frac{V}{K_t} \end{gathered} \tag{4}$$

The driving voltage of the motor is usually limited to some maximum value V_{max}. Because of this, the maximum speed at which the motor can be run is limited by the point at which the voltage drop across the motor is equal to the driving voltage, so that no net current flows across the terminals:

$$\omega_{max} = \frac{V_{max}}{K_t} \tag{5}$$

This simple conservation argument demonstrates the relationship of the torque constant to the maximum motor speed. We could have arrived at this result by directly calculating the voltage induced across the terminals using Lenz's law. However, our approach has the advantage that we do not need to look at the motor's design at all; we can base our conclusions entirely on data available in the data sheet.

The Terminal Resistance, R_m

In a motor driven with a constant voltage, the current flowing through the motor is determined by the back-emf (electro-motive force) voltage through torque constant, as described in (4), and by the terminal resistance of the motor. In general, the resistance and the induced voltage are treated as two voltage drops in series, as shown in Fig. 2. The terminal resistance can therefore be used to describe the torque produced at a constant voltage:

$$\begin{gathered} V = i \cdot R_m + \omega \cdot K_t \\ \tau = i \cdot K_t = \frac{K_t}{R_m}(V - \omega \cdot K_t) \end{gathered} \tag{6}$$

We can see that this agrees with the maximum speed derived in (5), because the speed at which no torque can be exerted on the rotor is equal to the driving voltage divided by the torque constant.

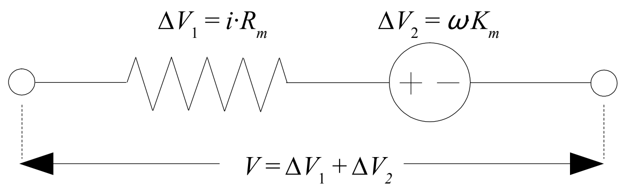

Figure 2: A low-frequency electrical schematic for a DC motor.

The Motor Inertia, I_m

The motor inertia I_m is the moment of inertia of the rotor. It is an important parameter defining characteristic of a motor because it defines the fastest transient response that the motor can have. If we model the acceleration of the rotor based on the constant voltage driving law, we can see this:

$$I_m \cdot \dot{\omega} = \frac{K_t}{R_m}(V - \omega \cdot K_t) \tag{7}$$

When we take the Laplace transform of this expression and rewrite it as a transfer function, we find:

$$\frac{\Omega(s)}{V(s)} = \frac{(K_t^{-1})}{(R_m \cdot J_m / K_t^2) \cdot s + 1} \tag{8}$$

This is a first order frequency response whose time constant increases linearly with I_m. This time constant $R_m \cdot I_m / K_t^2$ is often tabulated directly on the motor data sheet as the mechanical time constant of the motor, and it affects how quickly the motor can respond when driven at maximum voltage. Even if a large load is attached to a motor, the inertia of the motor may still dominate the mechanical response if the transmission ratio in the gearbox is large enough. You may recall that the effective inertia of an object on the motor side of a transmission is reduced by a factor of the gear ratio squared,

$$I_{effective} = I_m + \frac{I_{load}}{r^2} \tag{9}$$

Here the gear ratio r is defined as the ratio of motor torque to output torque. Thus, when using a motor with a large gear reduction ratio, like the 50:1 harmonic drives you will use later in the class, the effective inertia can still be dominated by the rotor inertia[3].

The Terminal Inductance, L_m

In order to understand how the terminal inductance is important to the specification of a motor, we must briefly explain how a pulse width-modulated (PWM) motor driver works. You may recall from 2.004 that you were using giant current amplifier boxes to drive your DC motor apparatus. These linear amplifiers produce a constant current, but they must be extremely bulky in order to be efficient. Most motors are driven instead with a high-frequency, switched voltage signal having some period T_{PWM}, as shown in Fig. 3. The motor current controlled by varying the width of the high-voltage pulse, T_{DC}.

3 When using a gearbox with a high reduction ratio, it is also a good idea to check the effective gearbox inertia, which is usually given on the gearbox data sheet. See chapter 2, page 4 in the notes for more on inertia modeling.

Without getting too heavily into the mathematics, we can explain how this can be used to obtain a steady output current by calculating the pulse train as a summed series of cosine waves,

$$V(t)=\frac{V_D \cdot T_{DC}}{T_{PWM}}+V_D\sum_{k=1}^{\infty} a_k \cos\left(\frac{t \cdot 2\pi \cdot k}{T_{PWM}}\right) \tag{10}$$

The coefficients a_k are Fourier series coefficients, whose value can be calculated using the cosine component of the discrete Fourier transform,

$$a_k=\frac{1}{T_{PWM}}\int_{-T_{DC}/2}^{T_{DC}/2} \cos\left(\frac{t \cdot 2\pi \cdot k}{T_{PWM}}\right) dt=\frac{1}{\pi \cdot k}\sin\left(\frac{\pi \cdot k \cdot T_{DC}}{T_{PWM}}\right) \tag{11}$$

Photo by ladyada on Flickr.

The Arduino motor shield
PWM amplifier we will use

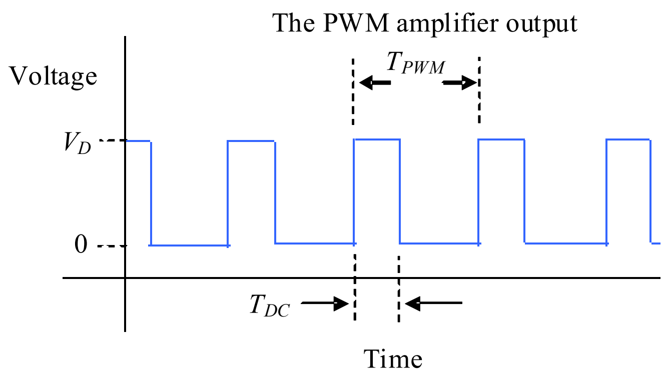

Figure 3: The voltage supplied by the Arduino motor shield is a square pulse train with period T_{PWM} and amplitude V_D. By varying the time that the voltage is high, T_{DC}, the current flowing through the motor can be controlled.

The goal of a PWM current amplifier is to get this high-frequency switching input to produce a smooth, constant current in the motor. To accomplish this, the inductance and resistance of the motor is used as a low-pass filter to filter out all of the high frequency components of (10). Figure 4 shows a high-frequency electrical motor model, including the terminal resistance, the induced back-emf voltage, and most importantly the *terminal inductance*. This model can be used to produce a frequency-domain description of the current response in the motor,

$$\begin{gathered} V(s)=(L_m \cdot s+R_m) \cdot I(s)+K_m \cdot \Omega(s) \\ I(s)=\frac{1/R_m}{(L_m/R_m) \cdot s+1} V(s)+\frac{K_m/R_m}{(L_m/R_m) \cdot s+1} \Omega(s) \end{gathered} \tag{12}$$

Focusing on the first term (the transfer function between driving voltage and current), we find that the motor windings act as a first-order low-pass filter whose time constant is equal to L_m/R_m,

$$\left|\frac{I(j \cdot \omega)}{V(j \cdot \omega)}\right|=\frac{1/R_m}{\sqrt{(L_m/R_m)^2 \cdot \omega^2+1}} \tag{13}$$

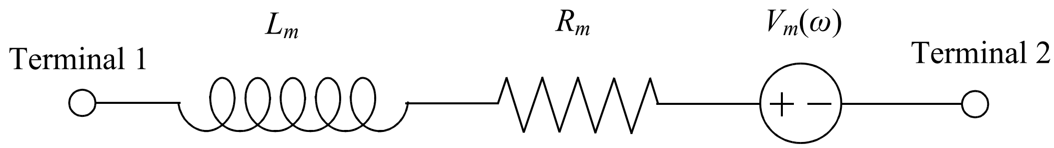

Figure 4: A high-frequency electrical model of a DC motor. The motor inductance is important at high frequencies because it is used to smooth out the current response to a switched voltage input.

Often, manufacturers will calculate L_m/R_m and list it on the data sheet as the motor's *reactance* (electrical time constant), T_e. If the switching frequency $2\pi/T_{PWM}$ is much greater than the filter's low pass cut-off frequency $1/T_e$, then the high-frequency terms in the motor current response die away:

$$i(t) = \frac{V_D \cdot T_{DC}}{T_{PWM} \cdot R_m} + \frac{V_D}{R_m} \sum_{k=1}^{\infty} \frac{a_k}{\sqrt{(2\pi \cdot k \cdot L_m)^2 / (R_m T_{PWM})^2 + 1}} \cos\left(\frac{t \cdot 2\pi \cdot k}{T_{PWM}}\right) \tag{14}$$

Because the Fourier coefficients a_k calculated in (11) already die away as $1/k$, the amplitude of each high-frequency term drops off as $1/k^2$, leaving only the leading term. Figure 5 shows a spectrum plot of the power (squared current amplitude) at every frequency for a PWM circuit whose switching frequency is 10 times the low-pass filter frequency of the motor. You can see that the first high-frequency term in the current has less than 1% percent of the power of the low-frequency response. This leads to lower current dissipation in the motor and more efficient motor operation. As a motor designer, you are responsible for making sure that your PWM current amplifiers have a switching frequency that is sufficiently higher than the filter frequency determined by the motor's electrical time constant, L_m/R_m.

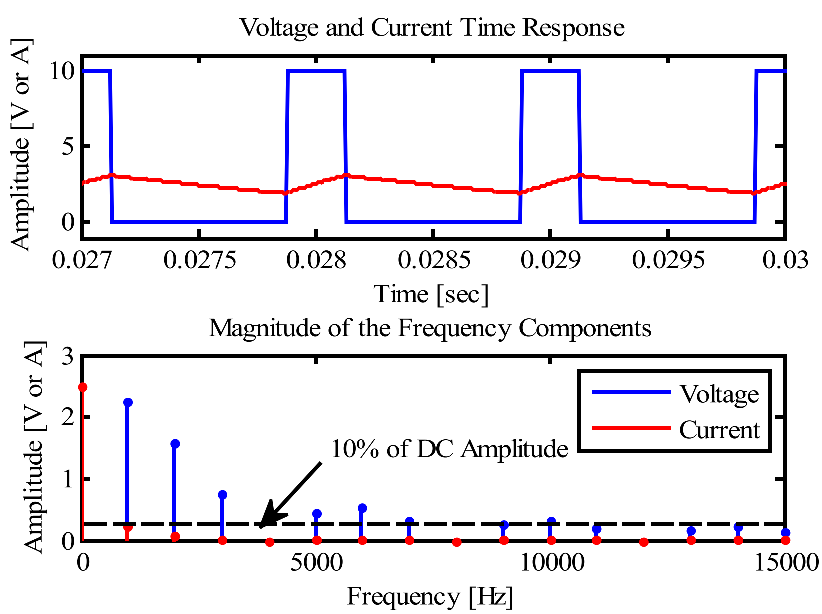

Figure 5: A plot of the amplitude for each frequency component in the pulse width-modulated motor voltage and current. The higher frequencies are quickly filtered out by the motor inductance, leaving a smoothed current signal. In this case, the switching frequency was 10 times the roll-off frequency of the motor's current response.

Encoders

Another crucial part of a servo motor which you should understand is the quadrature encoder, a device used to measure the output angle of the motor shaft for feedback control. You should be familiar with encoders from 2.004, but just in case, here's a brief review. An encoder is essentially a high-resolution mechanical counting device. Generally, a shaft encoder consists of a thin disc affixed to the motor shaft, inscribed with many small lines. These lines are read by two photo-detectors placed at the edge of the disc, providing output signals ChA and ChB. Figure 6 shows an exploded view of the optical encoders we will be using in this lab.

Figure 6: An exploded view of the Maxon Enc 22 encoder and the 2 channel signal it produces. Based on the signal on the other channel, a rising or falling edge from an encoder channel causes an increment or a decrement of the encoder count.

As the motor rotates, the encoder lines interrupt the photo-detectors to produce a square waves on ChA and ChB. The frequency of these square waves can be used to estimate the rate of motor rotation. In order to determine the direction of rotation, the sensor for ChB is placed at a 90 degree phase difference from ChA, as shown in Fig. 6. When a rising or falling edge is detected on ChA or ChB, the value of other channel can be used to determine the direction the encoder is traveling. If, for example, ChA sees a rising edge and the value of ChB is low, then ChA is leading ChB and the motor is moving in the positive direction. Conversely, if ChB sees a rising edge and ChA is low, then ChB is leading ChA and the motor is moving in the negative direction. In order to compute the angular position of the motor shaft, a typical encoder interface employs simple count-up, count-down logic, shown in the table below:

Table 2: The Counting Rules for Quadrature Encoders

Channel A State	Channel B State	Action Taken
Rising	Low	Count = Count + 1
High	Rising	Count = Count + 1
Falling	High	Count = Count + 1
Low	Falling	Count = Count + 1
Low	Rising	Count = Count - 1
Rising	High	Count = Count - 1
High	Falling	Count = Count - 1
Falling	Low	Count = Count - 1

In the laboratory, we will use an arduino microcontroller to interface to the encoder and integrate the encoder counts for use in feedback control of the motor.

Issues with Encoder Resolution

The number of lines on an encoder varies based on the encoder's quality (and price!). From Table 2 and Figure 6 above, you can see that for every single line on the encoder, 4 events are counted, corresponding to the rising and falling edges of both ChA and ChB. Encoders are typically specified in terms of the number of lines per revolution of the motor shaft. Keep in mind, though, that you get effectively 4 times that resolution in terms of encoder counts per revolution.

The resolution of an encoder can have a marked effect on the motor's performance. Because it is a digital measurement device, an encoder does not produce a truly smooth measurement of position. This can be problematic for control systems, because the effect of the quantized signal is equivalent to a noise input to the control system. Figure 7 shows a motor moving at a constant angular velocity. Instead of a smooth line, this motor's encoder will feed a staircase-like signal back to the controller. In this case, the effective noise that has been added to the signal would be the sawtooth wave obtained by subtracting the two signals. This sawtooth wave contains many high-frequency features that could introduce instability in the control system, or produce a high-pitched whine when the current resulting from the noise is fed back into the motor coils. This is especially bad if the control designer is attempting to take a derivative of the quantized encoder signal.

To avoid the problems resulting from encoder quantization, one could place a low-pass filter on the encoder signal to smooth out the sharp edges of the staircase signal. However, the additional poles resulting from this low-pass filter may adversely affect performance. In order to ensure that they do not, the encoder resolution could also be increased, reducing the size of the steps on the staircase signal. This would have the net effect of increasing the frequency of the encoder noise, so that the noise filter poles can be placed far enough away from the origin that they have minimal impact on the performance of the controller.

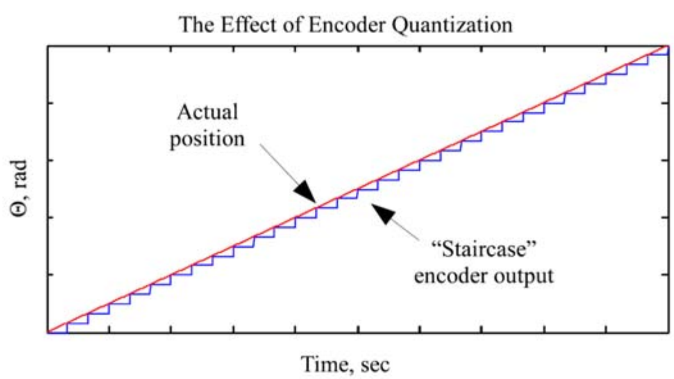

Figure 7: The effect of quantization on the position measurement of the motor.