8.334 Statistical Mechanics II: Statistical Physics of Fields
Spring 2008

IV.F Perturbative RG (Second Order)

The coarse grained Hamiltonian at second order in $\mathcal{U}$ is

$$\beta\tilde{\mathcal{H}}[\tilde{\vec{m}}] = V\delta f_b^0 + \int_0^{\Lambda/b} \frac{d^d\mathbf{q}}{(2\pi)^d} \left(\frac{t + Kq^2}{2}\right) |\tilde{m}(\mathbf{q})|^2 + \langle\mathcal{U}\rangle_\sigma - \frac{1}{2}\left(\langle\mathcal{U}^2\rangle_\sigma - \langle\mathcal{U}\rangle_\sigma^2\right) + O(\mathcal{U}^3).$$

(IV.47)

To calculate $\left(\langle\mathcal{U}^2\rangle_\sigma - \langle\mathcal{U}\rangle_\sigma^2\right)$ we need to consider all possible decompositions of two $\mathcal{U}$s into $\tilde{\vec{m}}$ and $\vec{\sigma}$ as in eq.(IV.34). Since each $\mathcal{U}$ can be broken up into 6 types of terms as in eq.(IV.35), there are 36 such possibilities for two $\mathcal{U}$s which can be arranged in a 6×6 table. Many of the elements of this table are either zero, or can be neglected at this stage, due to a number of considerations:

(i) All the 11 terms involving at least one factor of type [1] are zero because they cannot be contracted into a *connected* piece, and the disconnected elements cancel in calculating the cumulant.

(ii) An additional 12 terms (such as $[2] \times [3]$) involve an *odd* number of $\vec{\sigma}$s and are zero due to their *parity*.

(iii) Two terms, $[2]\times[5]$ and $[5]\times[2]$, involve a vertex where two $\vec{\sigma}$s are contracted together, leaving a $\tilde{m}(\mathbf{q}^<)$ and a $\vec{\sigma}(\mathbf{q}^>)$. This configuration is not allowed by the δ–function which ensures momentum conservation for the vertex, as by construction $\mathbf{q}^> + \mathbf{q}^< \neq \mathbf{0}$.

(iv) Terms $[3] \times [6]$, $[4] \times [6]$ and their partners by exchange have two factors of $\tilde{\vec{m}}$. They involve *two loop* integrations, and appear as corrections to the coefficient $\tilde{t}$. We shall denote their net effect by A, which as noted earlier does not need to be known precisely at this order.

(v) The term $[5] \times [5]$ also involves two factors of $\tilde{\vec{m}}$, while $[2] \times [2]$ includes 6 such factors. The latter is important as it indicates that the space of parameters *is not closed* at this order. Even if initially zero, a term proportional to m^6 is generated under RG. In fact, considerations of momentum conservation indicate that both these terms are zero for $\mathbf{q} = 0$, and are thus contributions to q^2m^2 and q^2m^6. We shall comment on their effect later on.

(vi) The contributions resulting from $[6]\times[6]$ are constants, and will be collectively denoted by $u^2 V\delta f_b^2$.

(vii) The terms $[3] \times [3]$, $[3] \times [4]$, $[4] \times [3]$, and $[4] \times [4]$ contribute to $\tilde{m}^4$. For example, $[3] \times [3]$ results in

$$
\frac{u^2}{2} \times 2 \times 2 \times 2 \int_0^{\Lambda/b} \frac{d^d\mathbf{q}_1 \cdots d^d\mathbf{q}_4}{(2\pi)^{4d}} \int_{\Lambda/b}^{\Lambda} \frac{d^d\mathbf{k}_1 d^d\mathbf{k}_2 d^d\mathbf{k}_1' d^d\mathbf{k}_2'}{(2\pi)^{4d}}
$$
$$
\times (2\pi)^{2d}\delta^d(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{k}_1 + \mathbf{k}_2)\delta^d(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{q}_3 + \mathbf{q}_4)
$$
$$
\times \frac{\delta_{\alpha\alpha'}(2\pi)^d\delta^d(\mathbf{k}_1 + \mathbf{k}_1')}{t + Kk_1^{2'}}\frac{\delta_{\alpha\alpha'}(2\pi)^d\delta^d(\mathbf{k}_2 + \mathbf{k}_2')}{t + Kk_2^{2'}}\tilde{m}(\mathbf{q}_1)\cdot\tilde{m}(\mathbf{q}_2)\tilde{m}(\mathbf{q}_3)\cdot\tilde{m}(\mathbf{q}_4) =
$$
$$
4nu^2 \int_0^{\Lambda/b} \frac{d^d\mathbf{q}_1 \cdots d^d\mathbf{q}_4}{(2\pi)^{4d}}(2\pi)^d\delta^d(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3 + \mathbf{q}_4)\tilde{m}(\mathbf{q}_1)\cdot\tilde{m}(\mathbf{q}_2)\tilde{m}(\mathbf{q}_3)\cdot\tilde{m}(\mathbf{q}_4)
$$
$$
\times \int \frac{d^d\mathbf{k}}{(2\pi)^d}\frac{1}{(t + Kk^2)\left(t + K(\mathbf{q}_1 + \mathbf{q}_2 - \mathbf{k})^2\right)}.
$$
$$\text{(IV.48)}$$

The contractions from terms $[3] \times [4]$, $[4] \times [3]$, and $[4] \times [4]$ lead to similar expressions with prefactors of 8, 8, and 16 respectively. Apart from the dependence on $\mathbf{q}_1$ and $\mathbf{q}_2$, the final result has the form of $\mathcal{U}[\tilde{m}]$. In fact the last integral can be expanded as

$$
f(\mathbf{q}_1 + \mathbf{q}_2) = \int \frac{d^d\mathbf{k}}{(2\pi)^d}\frac{1}{(t + Kk^2)^2}\left[1 - \frac{2K\mathbf{k}\cdot(\mathbf{q}_1 + \mathbf{q}_2) - K(\mathbf{q}_1 + \mathbf{q}_2)^2}{(t + Kk^2)} + \cdots\right]. \quad \text{(IV.49)}
$$

After fourier transforming back to real space we find in addition to m^4, such terms as $m^2(\nabla m)^2, m^2\nabla^2 m^2, \cdots$.

Putting all contributions together, the coarse grained Hamiltonian at order of u^2 takes the form

$$
\beta\tilde{\mathcal{H}} = V\left(\delta f_b^0 + u\delta f_b^1 + u^2\delta f_b^2\right)
$$
$$
+ \int_0^{\Lambda/b} \frac{d^d\mathbf{q}}{(2\pi)^d}|\tilde{m}(\mathbf{q})|^2\left[\frac{t + Kq^2}{2} + 2u(n+2)\int_{\Lambda/b}^{\Lambda}\frac{d^d\mathbf{k}}{(2\pi)^d}\frac{1}{t + Kk^2} - \frac{u^2}{2}A(t, K, q^2)\right]
$$
$$
+ \int_0^{\Lambda/b} \frac{d^d\mathbf{q}_1 \cdots d^d\mathbf{q}_4}{(2\pi)^{4d}}\tilde{m}(\mathbf{q}_1)\cdot\tilde{m}(\mathbf{q}_2)\tilde{m}(\mathbf{q}_3)\cdot\tilde{m}(\mathbf{q}_4)
$$
$$
\times \left[u - \frac{u^2}{2}(8n + 64)\int_{\Lambda/b}^{\Lambda}\frac{d^d\mathbf{k}}{(2\pi)^d}\frac{1}{(t + Kk^2)^2} + \mathcal{O}(u^2q^2)\right] + \mathcal{O}(u^2\tilde{m}^6q^2, \cdots) + \mathcal{O}(u^3).
$$
$$\text{(IV.50)}$$

IV.G The ϵ–Expansion

The parameter space (K, t, u) is no longer closed at this order; several new interactions proportional to m^2, m^4, and m^6, all consistent with symmetries of the problem, appear in

the coarse grained Hamiltonian at second order in u. Ignoring these interactions for the time being, the coarse grained parameters are given by

$$
\begin{cases}
\tilde{K} = K - u^2 A''(0) \\
\tilde{t} = t + 4(n+2)\, u \int_{\Lambda/b}^{\Lambda} \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{1}{t + Kk^2} - u^2 A(0) \\
\tilde{u} = u - 4(n+8)u^2 \int_{\Lambda/b}^{\Lambda} \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{1}{(t + Kk^2)^2}
\end{cases}
\quad , \qquad \text{(IV.51)}
$$

where $A(0)$ and $A''(0)$ correspond to the first two terms in the expansion of $A(t, K, q^2)$ in eq.(IV.50) in powers of q.

After the *rescaling* $\mathbf{q} = b^{-1}\mathbf{q}'$, and *renormalization* $\tilde{\vec{m}} = z\vec{m}'$, steps of the RG procedure, we obtain

$$
K' = b^{-d-2}z^2 \tilde{K}, \qquad t' = b^{-d}z^2 \tilde{t}, \qquad u' = b^{-3d}z^4 \tilde{u}. \qquad \text{(IV.52)}
$$

As before, the renormalization parameter z is chosen such that $K' = K$, leading to

$$
z^2 = \frac{b^{d+2}}{(1 - u^2 A''(0)/K)} = b^{d+2}\left(1 + O(u^2)\right). \qquad \text{(IV.53)}
$$

The value of z does depend on the fixed point position u^*. But as u^* is of the order of ϵ, $z = b^{1+\frac{d}{2}+\mathcal{O}(\epsilon^2)}$, is not changed at the lowest order. Using this value of z, and following the previous steps for constructing differential recursion relations, we obtain

$$
\begin{cases}
\dfrac{dt}{d\ell} = 2t + \dfrac{4u(n+2)K_d\Lambda^d}{t + K\Lambda^2} - A(t, K, \Lambda)u^2 \\[3mm]
\dfrac{du}{d\ell} = (4 - d)u - \dfrac{4(n+8)K_d\Lambda^d}{(t + K\Lambda^2)^2}u^2
\end{cases}
\qquad . \qquad \text{(IV.54)}
$$

The fixed points are obtained from $dt/d\ell = du/d\ell = 0$. In addition to the Gaussian fixed point at $u^* = t^* = 0$, discussed in the previous section, there is now a non-trivial fixed point located at

$$
\begin{cases}
u^* = \dfrac{(t^* + K\Lambda^2)^2}{4(n+8)K_d\Lambda^d}\epsilon = \dfrac{K^2}{4(n+8)K_4}\epsilon + \mathcal{O}(\epsilon^2) \\[3mm]
t^* = -\dfrac{2u^*(n+2)K_d\Lambda^d}{t^* + K\Lambda^2} = -\dfrac{(n+2)}{2(n+8)}K\Lambda^2\,\epsilon + \mathcal{O}(\epsilon^2)
\end{cases}
\qquad . \qquad \text{(IV.55)}
$$

The above expressions have been further simplified by systematically keeping terms to first order in $\epsilon = 4 - d$.

Linearizing the recursion relations in the vicinity of the fixed point results in

$$\frac{d}{d\ell}\begin{pmatrix}\delta t\\\delta u\end{pmatrix}=\begin{pmatrix}2-\frac{4(n+2)K_d\Lambda^d}{(t^*+K\Lambda^2)^2}u^*-A'u^{*2} & \frac{4(n+2)K_d\Lambda^d}{t^*+K\Lambda^2}-2Au^*\\\frac{8(n+8)K_d\Lambda^d}{(t^*+K\Lambda^2)^3}u^{*2} & \epsilon-\frac{8(n+8)K_d\Lambda^d}{(t^*+K\Lambda^2)^2}u^*\end{pmatrix}\begin{pmatrix}\delta t\\\delta u\end{pmatrix}. \tag{IV.56}$$

At the Gaussian fixed point, $t^*=u^*=0$, eq.(IV.45) is reproduced. At the new fixed point of eqs.(IV.55),

$$\frac{d}{d\ell}\begin{pmatrix}\delta t\\\delta u\end{pmatrix}=\begin{pmatrix}2-\frac{4(n+2)K_4\Lambda^4}{K^2\Lambda^4}\frac{K^2\epsilon}{4(n+8)K_4} & \cdots\cdots\\\mathcal{O}(\epsilon^2) & \epsilon-\frac{8(n+8)K_4\Lambda^4}{K^2\Lambda^4}\frac{K^2\epsilon}{4(n+8)K_4}\end{pmatrix}\begin{pmatrix}\delta t\\\delta u\end{pmatrix}. \tag{IV.57}$$

We have not explicitly calculated the top element of the second column as it is not necessary for calculating the eigenvalues. This is because the lower element of the first column is zero to order of ϵ. Hence the eigenvalues are determined by the diagonal elements alone. The first eigenvalue is positive, controlling the instability of the fixed point,

$$y_t=2-\frac{(n+2)}{(n+8)}\epsilon+\mathcal{O}(\epsilon^2). \tag{IV.58}$$

The second eigenvalue,

$$y_u=-\epsilon+\mathcal{O}(\epsilon^2), \tag{IV.59}$$

is negative for $d<4$. The new fixed point thus has co-dimension of one and can describe the phase transition in these dimensions. It is quite satisfying that while various intermediate results, such as the position of the fixed point depend on such microscopic parameters as K and Λ, the final eigenvalues are pure numbers, only depending on n and $d=4-\epsilon$. These eigenvalues characterize the *universality classes* of rotational symmetry breaking in $d<4$. (Long range interaction may lead to new universality classes.)

The divergence of the correlation length, $\xi\sim(\delta t)^{-\nu}$, is controlled by the exponent

$$\nu=\frac{1}{y_t}=\left\{2\left[1-\frac{(n+2)}{2(n+8)}\epsilon\right]\right\}^{-1}=\frac{1}{2}+\frac{1}{4}\frac{n+2}{n+8}\epsilon+\mathcal{O}(\epsilon^2). \tag{IV.60}$$

The singular part of the free energy scales as, $f\sim(\delta t)^{2-\alpha}$, and the heat capacity diverges with the exponent

$$\alpha=2-d\nu=2-\frac{(4-\epsilon)}{2}\left[1+\frac{1}{2}\frac{n+2}{n+8}\epsilon\right]=\frac{4-n}{2(n+8)}\epsilon+\mathcal{O}(\epsilon^2). \tag{IV.61}$$

To complete the calculation of critical exponents, we need the eigenvalue associated with the (relevant) symmetry breaking field h. This is easily found by adding a term $-\vec{h}\cdot$

$\int d^d\mathbf{x}\, \vec{m}(\mathbf{x}) = -\vec{h} \cdot \vec{m}(\mathbf{q} = \mathbf{0})$ to the Hamiltonian. This term is not effected by coarse graining or rescaling, and after the renormalization step changes to $-z\vec{h} \cdot \vec{m}'(\mathbf{q}' = \mathbf{0})$, implying

$$h' = zh = b^{1+\frac{d}{2}}h, \implies y_h = 1 + \frac{d}{2} + \mathcal{O}(\epsilon^2) = 3 - \frac{\epsilon}{2} + \mathcal{O}(\epsilon^2). \tag{IV.62}$$

The vanishing of magnetization as $T \to T_c^-$, is controlled by the exponent

$$
\begin{aligned}
\beta &= \frac{d - y_h}{y_t} = \left(\frac{4-\epsilon}{2} - 1\right) \times \frac{1}{2}\left(1 + \frac{n+2}{2(n+8)}\epsilon + \mathcal{O}(\epsilon^2)\right) \\
&= \frac{1}{2} - \frac{3}{2(n+8)}\epsilon + \mathcal{O}(\epsilon^2),
\end{aligned}
\tag{IV.63}
$$

while the susceptibility diverges as $\chi \sim (\delta t)^{-\gamma}$, with

$$\gamma = \frac{2y_h - d}{y_t} = 2 \times \frac{1}{2}\left(1 + \frac{n+2}{2(n+8)}\epsilon\right) = 1 + \frac{n+2}{2(n+8)}\epsilon + O(\epsilon^2). \tag{IV.64}$$

Using the above results, we can estimate various exponents as a function of d and n. For example, for $n = 1$, by setting $\epsilon = 1$ or 2 in eqs.(IV.60) and (IV.63) we obtain the values $\nu(1) \approx 0.58$, $\nu(2) \approx 0.67$, and $\beta(1) \approx 0.33$, $\beta(2) \approx 0.17$. The best estimates of these exponents in $d = 3$ are $\nu \approx 0.63$, and $\beta \approx 0.32$. In $d = 2$ the exact values are known to be $\nu = 1$ and $\beta = 0.125$. The estimates for β are quite good, while those for ν are less reliable. It is important to note that in all cases these estimates are an improvement over the mean field (saddle point) values. Since the expansion is around four dimensions, the results are more reliable in $d = 3$ than in $d = 2$. In any case, they correctly describe the decrease of β with lowering dimension, and the increase of ν. They also correctly describe the trends with varying n at a fixed d as indicated by the following table of exponents $\alpha(n)$.

	$n = 1$	$n = 2$	$n = 3$	$n = 4$
$\mathcal{O}(\epsilon)$ at $\epsilon = 1$	0.17	0.11	0.06	0
Experiments in $d = 3$	0.11	-0.01	-0.12	–

Although the sign of α is incorrectly predicted at this order for $n = 2$ and 3, the decrease of α with increasing n is correctly described.