8.334 Statistical Mechanics II: Statistical Physics of Fields
Spring 2008

III.D The Renormalization Group (Conceptual)

Success of the scaling theory in correctly predicting various exponent identities strongly supports the assumption that close to the critical point the correlation length ξ, is the only important length scale, and that microscopic length scales are irrelevant. The critical behavior is dominated by fluctuations that are self–similar up to the scale ξ. The self–similarity is of course only statistical, in that a magnetization configuration is generated with a weight $W[\vec{m}(\mathbf{x})] \propto \exp\{-\beta\mathcal{H}[\vec{m}(\mathbf{x})]\}$. Kadanoff suggested taking advantage of the self–similarity of the fluctuations to gradually eliminate the correlated degrees of freedom at length scales $x \ll \xi$, until one is left with the relatively simple, uncorrelated degrees of freedom at scale ξ. This is achieved through a procedure called the *renormalization group* (RG), whose conceptual foundation is the three steps outlined in this section.

(1) *Coarse Grain:* There is an implicit short distance length scale a, for allowed variations of $\vec{m}(\mathbf{x})$ in the system. This is the lattice spacing for a model of spins, or the coarse graining scale that underlies the Landau–Ginzburg Hamiltonian. In a digital picture of the system, a corresponds to the pixel size. The first step of the RG is to decrease the resolution by changing this minimum scale to ba ($b > 1$). The *coarse–grained* magnetization is then given by

$$\overline{m}_i(\mathbf{x}) = \frac{1}{b^d} \int_{\text{Cell centered at } \mathbf{x}} d^d\mathbf{x}' m_i(\mathbf{x}'). \tag{III.27}$$

(2) *Rescale:* Due to the change in resolution, the coarse grained 'picture' is grainier than the original. The original resolution of a can be restored by decreasing all length scales by a factor of b, i.e. by setting

$$\mathbf{x}_{\text{new}} = \frac{\mathbf{x}_{\text{old}}}{b}. \tag{III.28}$$

(3) *Renormalize:* The variations of fluctuations in the rescaled magnetization profile is in general different from the original, i.e. there is a difference in contrast between the pictures. This can be remedied by introducing a change of contrast by a factor ζ, through defining a *renormalized* magnetization

$$\vec{m}_{\text{new}}(\mathbf{x}_{\text{new}}) = \frac{1}{\zeta b^d} \int_{\text{Cell centered at } b\mathbf{x}_{\text{new}}} d^d\mathbf{x}' \vec{m}(\mathbf{x}'). \tag{III.29}$$

By following these steps, for each configuration $\vec{m}_{\text{old}}(\mathbf{x})$, we generate a renormalized configuration $\vec{m}_{\text{new}}(\mathbf{x})$. Eq.(III.29) can be regarded as a mapping from one set of random variables to another, and can be used to construct the probability distribution, or weight

$W_b[\vec{m}_{\rm new}(\mathbf{x})] \equiv \exp\{-\beta\mathcal{H}_b[\vec{m}_{\rm new}(\mathbf{x})]\}$. Kadanoff's insight was that since on length scales less than ξ, the renormalized configurations are statistically similar to the original ones, they may be distributed by a Hamiltonian $\beta\mathcal{H}_b$ that is also 'close' to the original. In particular, the original Hamiltonian becomes critical by tuning the two parameter t and h to zero, at which point the dominant configurations are similar to those of the rescaled system; the critical Hamiltonian is thus invariant under such rescaling. In the original problem, one moves away from criticality for finite t and h. Kadanoff's assumption is that the corresponding new Hamiltonian is also described by non-zero $t_{\rm new}$ or $h_{\rm new}$.

The assumption that the vicinity of the original and renormalized Hamiltonians to criticality is described by the two parameters t and h greatly simplifies the analysis. The effect of the RG transformation on the probability of configurations is now described by the two parameter mappings $t_{\rm new} \equiv t_b(t_{\rm old}, h_{\rm old})$ and $h_{\rm new} \equiv h_b(t_{\rm old}, h_{\rm old})$. The next assumption is that since the transformation only involves changes at the shortest length scales, it cannot cause any singularities. The renormalized parameters must be *analytic* functions of the original ones, and hence expandable as

$$\begin{cases} t_b(t,h) = A(b)t + B(b)h + \cdots \\ h_b(t,h) = C(b)t + D(b)h + \cdots \end{cases} . \tag{III.30}$$

Note that there are no constant terms in the above Taylor expansions. This expresses the condition that if $\beta\mathcal{H}$ is at its critical point ($t = h = 0$), then $\beta\mathcal{H}_b$ is also at criticality, and $t_{\rm new} = h_{\rm new} = 0$. Furthermore, due to rotational symmetry, under the combined transformation $(m(x) \mapsto -m(x), h \mapsto -h, t \mapsto t)$ the weight of a configuration is unchanged. As this symmetry is preserved by the RG, the coefficients B and C in the above expression must be zero, leading to the further simplifications

$$\begin{cases} t_b(t,h) = A(b)t + \cdots \\ h_b(t,h) = D(b)h + \cdots \end{cases} . \tag{III.31}$$

The remaining coefficients $A(b)$ and $D(b)$ depend on the (arbitrary) rescaling factor b, and trivially $A(1) = D(1) = 1$ for $b = 1$. Since the above transformations can be carried out in sequence, and the net effect of rescalings of b_1 and b_2 is a change of scale by $b_1 b_2$, the RG procedure is sometimes referred to as a *semi-group*. The term applies to the action of RG on the space of configurations: each magnetization profile is mapped uniquely to one at larger scale, but the inverse process is non-unique as some short scale information is lost in the coarse graining. (There is in fact no problem with inverting the

transformation in the space of the parameters of the Hamiltonian.) The dependence of A and D in eqs.(III.31) on b can be deduced from this group property. Since at $b = 1$, $A = D = 1$, and $t(b_1 b_2) \approx A(b_1)A(b_2)t \approx A(b_1 b_2)t$; we must have $A(b) = b^{y_t}$, and similarly $D(b) = b^{y_h}$, yielding

$$\begin{cases} t' \equiv t_b = b^{y_t} t + \cdots \\ h' \equiv h_b = b^{y_h} h + \cdots \end{cases} . \tag{III.32}$$

If $\beta\mathcal{H}_{\rm old}$ is slightly away from criticality, it is described by a large but finite correlation length $\xi_{\rm old}$. After the RG transformation, due to the rescaling in eq.(III.28), the new correlation length is smaller by a factor of b. Hence the renormalized Hamiltonian is less critical, and the RG procedure moves the parameters further away from the origin, i.e. the exponents y_t and y_h must be positive.

- We can now explore some consequences of the assumptions leading to eq.(III.32).

1. *The free energy:* The RG transformation is a many to one map of the original configurations to new ones. Since the weight of a new configuration, $W'([m'])$, is the sum of the weights $W([m])$, of old configurations, the partition function is preserved, i.e.

$$Z = \int DmW([m]) = \int Dm'W'([m']) = Z'. \tag{III.33}$$

Hence $\ln Z = \ln Z'$, and the corresponding free energies are related by

$$Vf(t,h) = V'f(t',h'). \tag{III.34}$$

In d–dimensions, the rescaled volume is smaller by a factor of b^d, and

$$f(t,h) = b^{-d} f(b^{y_t} t, b^{y_t} h), \tag{III.35}$$

where we have made use of the assumption that two free energies are obtained from the *same Hamiltonian* in which only the parameters t and h have changed according to eqs.(III.32). Eq.(III.35) describes a *homogeneous* function of t and h. This is made apparent by choosing a rescaling factor b such that b^{y_t} is a constant, say unity, i.e. $b = t^{-1/y_t}$, leading to

$$f(t,h) = t^{d/y_t} f(1, h/t^{y_h/y_t}) \equiv t^{d/y_t} g_f(h/t^{y_h/y_t}). \tag{III.36}$$

We have thus recovered the scaling form in eq.(III.4), and can identify the exponents as

$$2 - \alpha = d/y_t \qquad , \qquad \Delta = y_h/y_t. \tag{III.37}$$

2. *Correlation length:* All length scales are reduced by a factor of b during the RG transformation. This is also true of the correlation length, $\xi' = \xi/b$, implying

$$\xi(t,h) = b\xi(b^{y_t}t, b^{y_h}h) = t^{-1/y_t}\xi(1, h/t^{y_h/y_t}) \sim t^{-\nu}. \tag{III.38}$$

This identifies $\nu = 1/y_t$, and using eq.(III.37) the hyperscaling identity, $2 - \alpha = d\nu$, is recovered.

3. *Magnetization:* From the homogenous form of the free energy (eq.(III.36)), we can obtain other bulk quantities such as magnetization. Alternatively, from the RG results for Z, V, and h, we may directly conclude

$$m(t,h) = -\frac{1}{V}\frac{\partial \ln Z(t,h)}{\partial h} = -\frac{1}{b^d V'}\frac{\partial \ln Z'(t',h')}{b^{-y_h}\partial h'} = b^{y_h - d} m(b^{y_t}t, b^{y_h}h). \tag{III.39}$$

Choosing $b = t^{-1/y_t}$, we obtain $\beta = (y_h - d)/y_t$, and $\Delta = y_h/y_t$ as before.

It is thus apparent that quite generally, the singular part of any quantity X has a homogeneous form

$$X(t,h) = b^{y_X} X(b^{y_t}t, b^{y_h}h). \tag{III.40}$$

For any conjugate pair of variables, contributing a term $\int d^d\mathbf{x} F \cdot X$, to the Hamiltonian, the *scaling dimensions* are related by $y_X = y_F - d$, where $F' = b^{y_F}F$ under RG.

III.E The Renormalization Group (Formal)

In the previous section we noted that all critical properties can be obtained from the recursion relations in eqs.(III.32). Though conceptually appealing, it is not clear how such a procedure can be formally carried out. In particular, why should the forms of the two Hamiltonians be identical, and why are two parameters t and h sufficient to describe the transformation? In this section we outline a more formal procedure for identifying the effects of the dilation operation on the Hamiltonian. The various steps of the program are as follows:

(1) Start with most general Hamiltonian allowed by symmetries. For example, in the presence of rotational symmetry,

$$\beta\mathcal{H} = \int d^d\mathbf{x} \left[\frac{t}{2}m^2 + um^4 + vm^6 + \cdots + \frac{K}{2}(\nabla m)^2 + \frac{L}{2}(\nabla^2 m)^2 + \cdots\right]. \tag{III.41}$$

A particular system with such symmetry is therefore completely specified by a point in the (infinite–dimensional) parameter space $S \equiv (t, u, v, \cdots, K, L, \cdots)$.

(2) Apply the three steps of renormalization in configuration space: (i) Coarse grain by b; (ii) Rescale, $\mathbf{x}' = \mathbf{x}/b$; and (iii) Renormalize, $m' = m/\zeta$. This defines a change of variables,

$$m'(\mathbf{x}') = \frac{1}{\zeta b^d} \int_{\text{Cell of size } b \text{ centered at } b\mathbf{x}'} d^d\mathbf{x} m(\mathbf{x}). \tag{III.42}$$

Given the probabilities $\mathcal{P}[m(\mathbf{x})] \propto \exp(-\beta\mathcal{H}[m(\mathbf{x})])$, for the original configurations, we can use the above change of variable to construct the corresponding probabilities $\mathcal{P}'[m'(\mathbf{x}')]$, for the new configurations. Naturally this is the most difficult step in the program.

(3) Since rotational symmetry is preserved by the RG procedure, the rescaled Hamiltonian must also be described by a point in the parameter space of eq.(III.41), i.e.

$$\begin{aligned} &\beta\mathcal{H}'[m'(\mathbf{x}')] \equiv \ln \mathcal{P}[m'(\mathbf{x}')] \\ &= f_b + \int d^d\mathbf{x}' \left[\frac{t'}{2} m'^2 + u' m'^4 + v' m'^6 + \cdots + \frac{K'}{2}(\nabla m')^2 + \frac{L'}{2}(\nabla^2 m')^2 + \cdots \right]. \end{aligned} \tag{III.43}$$

The renormalized parameters are functions of the original ones, i.e. $t' = t_b(t, u, ...)$; $u' = u_b(t, u, ...)$, etc., defining a mapping $S' = \Re_b S$ in parameter space.

(4) The operation $\Re_b$ describes the effects of dilation on the Hamiltonian of the system. Hamiltonians that describe statistically self–similar configurations must thus correspond to *fixed points* S^*, such that $\Re_b S^* = S^*$. Since the correlation length, a function of Hamiltonian parameters, is reduced by b under the RG operation (i.e. $\xi(S) = b\xi(\Re_b S)$), the correlation length at a fixed point must be zero or infinity. Fixed points with $\xi^* = 0$ describe independent fluctuations at each point and correspond to complete disorder (infinite temperature), or complete order (zero temperature). A fixed point with $\xi^* = \infty$ describes a critical point ($T = T_c$).

(5) Eqs.(III.32) represent a simplified case in which the parameter space is two dimensional. The point $t = h = 0$ is a fixed point, and the lowest order terms in these equations describe the behavior in its neighborhood. In general, we can study the stability of a fixed point by *linearizing* the recursion relations in its vicinity: Under RG, a point $S^* + \delta S$ is transformed to

$$S^*_\alpha + \delta S'_\alpha = S^*_\alpha + (\Re^L_b)_{\alpha\beta} \delta S_\beta + \cdots, \qquad \text{where} \qquad (\Re^L_b)_{\alpha\beta} \equiv \left. \frac{\partial S'_\alpha}{\partial S_\beta} \right|_{S^*}. \tag{III.44}$$

We now diagonalize the matrix $(\Re_b^L)_{\alpha\beta}$ to get the eigenvectors $\mathcal{O}_i$, and corresponding eigenvalues $\lambda(b)_i$. Because of the group property[†],

$$\Re_b^L \Re_{b'}^L \mathcal{O}_i = \lambda(b)_i \lambda(b')_i \mathcal{O}_i = \Re_{bb'}^L \mathcal{O}_i = \lambda(bb')_i \mathcal{O}_i. \tag{III.45}$$

Together with the condition $\lambda(1)_i = 1$, the above equation implies

$$\lambda(b)_i = b^{y_i}. \tag{III.46}$$

The vectors $\mathcal{O}_i$ are called scaling directions associated with the fixed point S^*, and y_i are the corresponding *anomalous dimensions*. Any Hamiltonian in the vicinity of the fixed point is described by a point $S = S^* + \Sigma_i g_i \mathcal{O}_i$. The renormalized Hamiltonian has to interaction parameters $S' = S^* + \Sigma_i g_i b^{y_i} \mathcal{O}_i$. The following terminology is used to classify the eigen-operators:

- If $y_i > 0$, g_i *increases* under scaling, and $\mathcal{O}_i$ is a *relevant operator*.
- If $y_i < 0$, g_i *decreases* under scaling, and $\mathcal{O}_i$ is an *irrelevant operator*.
- If $y_i = 0$, g_i is called a *marginal operator*, and higher order terms are necessary to track its behavior.

The subspace spanned by the irrelevant operators is the *basin of attraction* of the fixed point S^*. Since ξ always decreases under RG, and $\xi(S^*) = \infty$; then ξ is also infinite for any point on the basin of attraction of S^*. For a general point in the vicinity of S^*, the correlation length satisfies

$$\xi(g_1, g_2, \cdots) = b\xi(b^{y_1} g_1, b^{y_2} g_2, \cdots). \tag{III.47}$$

For a sufficiently large b, all the irrelevant operators scale to zero. The leading singularities of ξ are then determined by the remaining set of *relevant* operators. In particular if the operators are indexed in order of decreasing dimensions, we can choose b such that $b^{y_1} g_1 = 1$. In this case, eq.(III.47) implies

$$\xi(g_1, g_2, \cdots) = g_1^{-1/y_1} f(g_2/g_1^{y_2/y_1}, \cdots). \tag{III.48}$$

[†] The group property $\Re_b^L \Re_{b'}^L = \Re_{bb'}^L = \Re_{b'}^L \Re_b^L$, also implies that the linearized matrices for different b commute. It is thus possible to diagonalize them simultaneously, implying that the eigenvectors $\{\mathcal{O}_i\}$ are independent of b.

We have thus obtained an exponent $\nu_1 = 1/y_1$, for the divergence of ξ, and a generalized set of gap exponents $\Delta_\alpha = y_\alpha/y_1$, associated with g_α.

Let us imagine that the fixed point S^* describes the critical point of the magnet in eq.(III.41) at zero magnetic field. As the temperature, or some other control parameter, is changed, the coefficients of the Hamiltonian are altered, and the point S moves along a trajectory in parameter space. Except for a single point (at the critical temperature) the magnet has a finite correlation length. This can be achieved if the trajectory taken by the point S intersects the basis of attraction of S^* only at one point. To achieve this, the basin of attraction must have co-dimension one, i.e. the fixed point S^* must have one and only one relevant operator. This provides an explanation of *universality*, in that the very many microscopic details of the system make up the huge space of irrelevant operators comprising the basin of attraction. In the presence of a magnetic field, two system parameters must be adjusted to reach the critical point ($T = T_c$ and $h = 0$). Thus the magnetic field corresponds to an additional relevant operator at S^*. Again, other 'odd' interactions, such as $\{m^3, m^5, \cdots\}$ should not lead to any other relevant operators.

Although the formal procedure outlined in this section is quite rigorous, it suffers from some quite obvious shortcomings: How do we actually implement the RG transformations of step (2) analytically? There are an infinite number of interactions allowed by symmetry, and hence the space of parameters S, is inconveniently large. How do we know a priori that there are fixed points for the RG transformation; that $\Re_b$ can be linearized; that relevant operators are few, etc.? Following the initial formulation of RG by Kadanoff, there was a period of uncertainty until Wilson showed how these steps can be implemented (at least perturbatively) in the Landau–Ginzburg model.