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8.334 Statistical Mechanics II: Statistical Physics of Fields
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Scaling & Renormalization

1. The nonlinear σ model describes n component unit spins. As we shall demonstrate later, in $d = 2$ dimensions, the recursion relations for temperature T , and magnetic field h , are

$$\begin{cases} \frac{dT}{d\ell} = \frac{(n-2)}{2\pi} T^2 \\ \frac{dh}{d\ell} = 2h \end{cases} ,$$

- (a) How does the correlation length diverge as $T \rightarrow 0$?
- (b) Write down the singular form of the free energy as $T, h \rightarrow 0$.
- (c) How does the susceptibility χ , diverge as $T \rightarrow 0$ for $h = 0$?

2. *Coupled scalars:* Consider the Hamiltonian

$$\beta\mathcal{H} = \int d^d\mathbf{x} \left[\frac{t}{2}m^2 + \frac{K}{2}(\nabla m)^2 - hm + \frac{L}{2}(\nabla^2\phi)^2 + v\nabla m \cdot \nabla\phi \right],$$

coupling two one component fields m and ϕ .

- (a) Write $\beta\mathcal{H}$ in terms of the Fourier transforms $m(\mathbf{q})$ and $\phi(\mathbf{q})$.
- (b) Construct a renormalization group transformation as in the text, by rescaling distances such that $\mathbf{q}' = b\mathbf{q}$; and the fields such that $m'(\mathbf{q}') = \tilde{m}(\mathbf{q})/z$ and $\phi'(\mathbf{q}') = \tilde{\phi}(\mathbf{q})/y$. Do not evaluate the integrals that just contribute a constant additive term.
- (c) There is a fixed point such that $K' = K$ and $L' = L$. Find y_t , y_h and y_v at this fixed point.
- (d) The singular part of the free energy has a scaling from $f(t, h, v) = t^{2-\alpha}g(h/t^\Delta, v/t^\omega)$ for t, h, v close to zero. Find α , Δ , and ω .
- (e) There is another fixed point such that $t' = t$ and $L' = L$. What are the relevant operators at this fixed point, and how do they scale?

3. *Anisotropic criticality:* A number of materials, such as liquid crystals, are anisotropic and behave differently along distinct directions, which shall be denoted parallel and perpendicular, respectively. Let us assume that the d spatial dimensions are grouped into n

parallel directions \mathbf{x}_{\parallel} , and $d - n$ perpendicular directions \mathbf{x}_{\perp} . Consider a one-component field $m(\mathbf{x}_{\parallel}, \mathbf{x}_{\perp})$ subject to a Landau–Ginzburg Hamiltonian, $\beta\mathcal{H} = \beta\mathcal{H}_0 + U$, with

$$\beta\mathcal{H}_0 = \int d^n \mathbf{x}_{\parallel} d^{d-n} \mathbf{x}_{\perp} \left[\frac{K}{2} (\nabla_{\parallel} m)^2 + \frac{L}{2} (\nabla_{\perp}^2 m)^2 + \frac{t}{2} m^2 - h m \right],$$

and

$$U = u \int d^n \mathbf{x}_{\parallel} d^{d-n} \mathbf{x}_{\perp} m^4.$$

(Note that $\beta\mathcal{H}$ depends on the **first** gradient in the \mathbf{x}_{\parallel} directions, and on the **second** gradient in the \mathbf{x}_{\perp} directions.)

- (a) Write $\beta\mathcal{H}_0$ in terms of the Fourier transforms $m(\mathbf{q}_{\parallel}, \mathbf{q}_{\perp})$.
- (b) Construct a renormalization group transformation for $\beta\mathcal{H}_0$, by rescaling coordinates such that $\mathbf{q}'_{\parallel} = b \mathbf{q}_{\parallel}$ and $\mathbf{q}'_{\perp} = c \mathbf{q}_{\perp}$ and the field as $m'(\mathbf{q}') = m(\mathbf{q})/z$. Note that parallel and perpendicular directions are scaled differently. Write down the recursion relations for K , L , t , and h in terms of b , c , and z . (The exact shape of the Brillouin zone is immaterial at this stage, and you do not need to evaluate the integral that contributes an additive constant.)
- (c) Choose $c(b)$ and $z(b)$ such that $K' = K$ and $L' = L$. At the resulting fixed point calculate the eigenvalues y_t and y_h for the rescalings of t and h .
- (d) Write the relationship between the (singular parts of) free energies $f(t, h)$ and $f'(t', h')$ in the original and rescaled problems. Hence write the unperturbed free energy in the homogeneous form $f(t, h) = t^{2-\alpha} g_f(h/t^{\Delta})$, and identify the exponents α and Δ .
- (e) How does the unperturbed zero-field susceptibility $\chi(t, h = 0)$, diverge as $t \rightarrow 0$?
In the remainder of this problem set $h = 0$, and treat U as a perturbation.
- (f) In the unperturbed Hamiltonian calculate the expectation value $\langle m(q)m(q') \rangle_0$, and the corresponding susceptibility $\chi_0(q) = \langle |m_q|^2 \rangle_0$, where q stands for $(\mathbf{q}_{\parallel}, \mathbf{q}_{\perp})$.
- (g) Write the perturbation U , in terms of the normal modes $m(q)$.
- (h) Using RG, or any other method, find the upper critical dimension d_u , for validity of the Gaussian exponents.
- (i) Write down the expansion for $\langle m(q)m(q') \rangle$, to first order in U , and reduce the correction term to a product of two point expectation values.
- (j) Write down the expression for $\chi(q)$, in first order perturbation theory, and identify the transition point t_c at order of u . (Do not evaluate any integrals explicitly.)

4. *Long-range interactions* between spins can be described by adding a term

$$\int d^d \mathbf{x} \int d^d \mathbf{y} J(|\mathbf{x} - \mathbf{y}|) \vec{m}(\mathbf{x}) \cdot \vec{m}(\mathbf{y}),$$

to the usual Landau–Ginzburg Hamiltonian.

(a) Show that for $J(r) \propto 1/r^{d+\sigma}$, the Hamiltonian can be written as

$$\begin{aligned} \beta \mathcal{H} = & \int \frac{d^d \mathbf{q}}{(2\pi)^d} \frac{t + K_2 q^2 + K_\sigma q^\sigma + \cdots}{2} \vec{m}(\mathbf{q}) \cdot \vec{m}(-\mathbf{q}) \\ & + u \int \frac{d^d \mathbf{q}_1 d^d \mathbf{q}_2 d^d \mathbf{q}_3}{(2\pi)^{3d}} \vec{m}(\mathbf{q}_1) \cdot \vec{m}(\mathbf{q}_2) \vec{m}(\mathbf{q}_3) \cdot \vec{m}(-\mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3) \quad . \end{aligned}$$

(b) For $u = 0$, construct the recursion relations for (t, K_2, K_σ) and show that K_σ is irrelevant for $\sigma > 2$. What is the fixed Hamiltonian in this case?

(c) For $\sigma < 2$ and $u = 0$, show that the spin rescaling factor must be chosen such that $K'_\sigma = K_\sigma$, in which case K_2 is irrelevant. What is the fixed Hamiltonian now?

(d) For $\sigma < 2$, calculate the generalized Gaussian exponents ν , η , and γ from the recursion relations. Show that u is irrelevant, and hence the Gaussian results are valid, for $d > 2\sigma$.

(e) For $\sigma < 2$, use a perturbation expansion in u to construct the recursion relations for (t, K_σ, u) as in the text.

(f) For $d < 2\sigma$, calculate the critical exponents ν and η to first order in $\epsilon = d - 2\sigma$.

[See M.E. Fisher, S.-K. Ma and B.G. Nickel, Phys. Rev. Lett. **29**, 917 (1972).]

(g) What is the critical behavior if $J(r) \propto \exp(-r/a)$? Explain!
