8.334 Statistical Mechanics II: Statistical Physics of Fields
Spring 2008

VII. Continuous Spins at Low Temperatures

VII.A The non-linear σ-model

Previously we considered low temperature expansions for *discrete* spins (Ising, Potts, etc.), in which the low energy excitations are droplets of incorrect spin in a uniform background selected by broken symmetry. These excitations occur at small scales, and are easily described by graphs on the lattice. By contrast, for *continuous* spins, the lowest energy excitations are long-wavelength Goldstone modes, as discussed in section II.C. The thermal excitation of these modes destroys the long-range order in dimensions $d \leq 2$. For d close to 2, the critical temperature must be small, making low temperature expansions a viable tool for the study of critical phenomena. As we shall demonstrate next, such an approach requires keeping track of the interactions between Goldstone modes.

Consider unit n-component spins on the sites of a lattice, i.e.

$$\vec{s}(\mathbf{i}) = (s_1, s_2, \cdots, s_n), \quad \text{with} \quad |\vec{s}(\mathbf{i})|^2 = s_1^2 + \cdots + s_n^2 = 1. \tag{VII.1}$$

The usual nearest neighbor Hamiltonian can be written as

$$-\beta\mathcal{H} = K\sum_{\langle \mathbf{ij}\rangle} \vec{s}(\mathbf{i}) \cdot \vec{s}(\mathbf{j}) = K\sum_{\langle \mathbf{ij}\rangle}\left(1 - \frac{(\vec{s}(\mathbf{i}) - \vec{s}(\mathbf{j}))^2}{2}\right). \tag{VII.2}$$

At low temperatures, the fluctuations between neighboring spins are small and the difference in eq.(VII.2) can be replaced by a gradient. Assuming a unit lattice spacing,

$$-\beta\mathcal{H} = -\beta E_0 - \frac{K}{2}\int d^d\mathbf{x}\,(\nabla\vec{s}(\mathbf{x}))^2\,, \tag{VII.3}$$

where the discrete index $\mathbf{i}$ has been replaced by a continuous vector $\mathbf{x} \in \Re^d$. A cutoff of $\Lambda \approx \pi$ is thus implicit in eq.(VII.3). Ignoring the ground state energy, the partition function is

$$Z = \int \mathcal{D}\left[\vec{s}(\mathbf{x})\delta\left(s(\mathbf{x})^2 - 1\right)\right] e^{-\frac{K}{2}\int d^d\mathbf{x}(\nabla\vec{s})^2}. \tag{VII.4}$$

A possible ground state configuration is $\vec{s}(\mathbf{x}) = (0, \cdots, 1)$. There are $n-1$ Goldstone modes describing the transverse fluctuations. To examine the effects of these fluctuations close to zero temperature, set

$$\vec{s}(\mathbf{x}) = (\pi_1(\mathbf{x}), \cdots, \pi_{n-1}(\mathbf{x}), \sigma(\mathbf{x})) \equiv (\vec{\pi}(\mathbf{x}), \sigma(\mathbf{x}))\,, \tag{VII.5}$$

where $\vec{\pi}(\mathbf{x})$ is an $n-1$ component vector. The unit length of the spin fixes $\sigma(\mathbf{x})$ in terms of $\vec{\pi}(\mathbf{x})$. For each degree of freedom

$$\begin{aligned} \int d\vec{s}\,\delta(s^2-1) &= \int_{-\infty}^{\infty} d\vec{\pi}\,d\sigma\delta\left(\pi^2+\sigma^2-1\right) \\ &= \int_{-\infty}^{\infty} d\vec{\pi}\,d\sigma\delta\left[\left(\sigma-\sqrt{1-\pi^2}\right)\left(\sigma+\sqrt{1-\pi^2}\right)\right] = \int_{-\infty}^{\infty} \frac{d\vec{\pi}}{2\sqrt{1-\pi^2}}\ , \end{aligned} \tag{VII.6}$$

where we have used the identity $\delta(ax) = \delta(x)/|a|$. Using this result, the partition function in eq.(VII.4) can be written as

$$\begin{aligned} Z &\propto \int \frac{\mathcal{D}\vec{\pi}(\mathbf{x})}{\sqrt{1-\pi(\mathbf{x})^2}}\, e^{-\frac{K}{2}\int d^d\mathbf{x}\left[(\nabla\vec{\pi})^2+\left(\nabla\sqrt{1-\pi^2}\right)^2\right]} \\ &= \int \mathcal{D}\vec{\pi}(\mathbf{x}) \exp\left\{-\int d^d\mathbf{x}\left[\frac{K}{2}(\nabla\vec{\pi})^2+\frac{K}{2}\left(\nabla\sqrt{1-\pi^2}\right)^2+\frac{\rho}{2}\ln(1-\pi^2)\right]\right\}. \end{aligned} \tag{VII.7}$$

In going from the lattice to the continuum, we have introduced a density $\rho = N/V = 1/a^d$ of lattice points. For unit lattice spacing $\rho = 1$, but for the purpose of renormalization we shall keep an arbitrary ρ. Whereas the original Hamiltonian was quite simple, the one describing the Goldstone modes $\vec{\pi}(\mathbf{x})$, is rather complicated. In selecting a particular ground state, the rotational symmetry was broken. The nonlinear terms in eq.(VII.7) ensure that this symmetry is properly reflected when considering only $\vec{\pi}$.

We can expand the nonlinear terms for the effective Hamiltonian in powers of $\vec{\pi}(\mathbf{x})$, resulting in a series

$$\beta\mathcal{H}[\vec{\pi}(\mathbf{x})] = \beta\mathcal{H}_0 + \mathcal{U}_1 + \mathcal{U}_2 + \cdots, \tag{VII.8}$$

where

$$\beta\mathcal{H}_0 = \frac{K}{2}\int d^d\mathbf{x}(\nabla\vec{\pi})^2, \tag{VII.9}$$

describes independent Goldstone modes, while

$$\mathcal{U}_1 = \int d^d\mathbf{x}\left[\frac{K}{2}(\vec{\pi}\cdot\nabla\vec{\pi})^2 - \frac{\rho}{2}\pi^2\right], \tag{VII.10}$$

is the first order perturbation when the terms in the series are organized according to powers of $T = 1/K$. Since we expect fluctuations $\langle\pi^2\rangle \propto T$, $\beta\mathcal{H}_0$ is order of one, the two terms in $\mathcal{U}_1$ are order of T; remaining terms are order of T^2 and higher. In the language of Fourier modes,

$$
\begin{aligned}
\beta\mathcal{H}_0 =& \frac{K}{2}\int \frac{d^d\mathbf{q}}{(2\pi)^d}\, q^2\, |\vec{\pi}\,(\mathbf{q})|^2, \\
\mathcal{U}_1 =& -\frac{K}{2}\int \frac{d^d\mathbf{q}_1\, d^d\mathbf{q}_2\, d^d\mathbf{q}_3}{(2\pi)^{3d}} \pi_\alpha(\mathbf{q}_1)\, \pi_\alpha(\mathbf{q}_2)\, \pi_\beta(\mathbf{q}_3)\, \pi_\beta(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3)\, (\mathbf{q}_1\cdot\mathbf{q}_3) \\
& -\frac{\rho}{2}\int \frac{d^d\mathbf{q}}{(2\pi)^d}\, |\vec{\pi}\,(\mathbf{q})|^2.
\end{aligned}
\tag{VII.11}
$$

For the non-interacting (quadratic) theory, the correlation functions of the Goldstone modes are

$$
\langle \pi_\alpha(\mathbf{q})\pi_\beta(\mathbf{q}')\rangle_0 = \frac{\delta_{\alpha,\beta}(2\pi)^d\delta^d(\mathbf{q}+\mathbf{q}')}{Kq^2}. \tag{VII.12}
$$

The resulting fluctuations in real space behave as

$$
\left\langle \pi(\mathbf{x})^2\right\rangle_0 = \int \frac{d^d\mathbf{q}}{(2\pi)^d}\left\langle |\vec{\pi}\,(\mathbf{q})|^2\right\rangle_0 = \frac{(n-1)}{K}\int_{1/L}^{1/a}\frac{d^d\mathbf{q}}{(2\pi)^d}\frac{1}{q^2} = \frac{(n-1)}{K}\frac{K_d\left(a^{2-d}-L^{2-d}\right)}{(d-2)}. \tag{VII.13}
$$

For $d>2$ the fluctuations are indeed proportional to T. However, for $d\leq 2$ they diverge as $L\to\infty$. This is a consequence of the Mermin–Wagner theorem on the absence of long range order in $d\leq 2$. Polyakov (1975) argued that this implies a critical temperature $T_c\sim\mathcal{O}(d-2)$ for such systems, and that an RG expansion in powers of T may provide a systematic way to explore critical behavior close to two dimensions.

To construct a perturbative RG, consider a spherical Brillouin zone of radius Λ, and divide the modes as $\vec{\pi}\,(\mathbf{q}) = \vec{\pi}^{\,<}(\mathbf{q}) + \vec{\pi}^{\,>}(\mathbf{q})$. The modes $\vec{\pi}^{\,<}$ involve momenta $0<|\mathbf{q}|<\Lambda/b$, while we shall integrate over the short wavelength fluctuations $\vec{\pi}^{\,>}$ with momenta in the shell $\Lambda/b<|\mathbf{q}|<\Lambda$. To order of T, the coarse–grained Hamiltonian is given by

$$
\beta\tilde{\mathcal{H}}\left[\vec{\pi}^{\,<}\right] = V\delta f_b^0 + \beta\mathcal{H}_0\left[\vec{\pi}^{\,<}\right] + \left\langle \mathcal{U}_1\left[\vec{\pi}^{\,<}+\vec{\pi}^{\,>}\right]\right\rangle_0^> + \mathcal{O}(T^2), \tag{VII.14}
$$

where $\langle\ \rangle_0^>$ indicates averaging over $\vec{\pi}^{\,>}$. The term proportional to ρ in eq.(VII.11) results in two contributions, one is a constant addition to free energy (from $\left\langle(\pi^>)^2\right\rangle$), and the other is simply $\rho(\pi^<)^2$. (The cross terms proportional to $\vec{\pi}^{\,<}\cdot\vec{\pi}^{\,>}$ vanish by symmetry.) The quartic part of $\mathcal{U}_1$ generates 16 terms. Nontrivial contributions arise from products of two $\vec{\pi}^{\,<}$ and two $\vec{\pi}^{\,>}$. There are three types of such contributions; the first has the form

$$
\begin{aligned}
\langle \mathcal{U}_1^a\rangle_0^> = 2\times\frac{-K}{2}\int &\frac{d^d\mathbf{q}_1\, d^d\mathbf{q}_2\, d^d\mathbf{q}_3}{(2\pi)^{3d}}(\mathbf{q}_1\cdot\mathbf{q}_3) \\
&\left\langle \pi_\alpha^>(\mathbf{q}_1)\pi_\alpha^>(\mathbf{q}_2)\right\rangle_0^> \pi_\beta^<(\mathbf{q}_3)\pi_\beta^<(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3).
\end{aligned}
\tag{VII.15}
$$

The integral over the shell momentum $\mathbf{q}_1$ is odd and this contribution is zero. (Two similar vanishing terms arise from contractions with different indices α and β.) The next term is a renormalization of ρ, arising from

$$
\begin{aligned}
\left\langle \mathcal{U}_1^b \right\rangle_0^> =& -\frac{K}{2}\int \frac{d^d\mathbf{q}_1\, d^d\mathbf{q}_2\, d^d\mathbf{q}_3}{(2\pi)^{3d}}(\mathbf{q}_1\cdot\mathbf{q}_3) \\
&\left\langle \pi_\alpha^>(\mathbf{q}_1)\pi_\beta^>(\mathbf{q}_3)\right\rangle_0^> \pi_\alpha^<(\mathbf{q}_2)\pi_\beta^<(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3) \\
=&\frac{K}{2}\int_0^{\Lambda/b}\frac{d^d\mathbf{q}}{(2\pi)^d}|\vec{\pi}^{\,<}(\mathbf{q})|^2 \times \int_{\Lambda/b}^{\Lambda}\frac{d^d\mathbf{k}}{(2\pi)^d}\frac{k^2}{Kk^2} \\
=&\frac{\rho}{2}\int_0^{\Lambda/b}\frac{d^d\mathbf{q}}{(2\pi)^d}|\vec{\pi}^{\,<}(\mathbf{q})|^2 \times \left(1-b^{-d}\right).
\end{aligned}
\tag{VII.16}
$$

(Note that in general $\rho = N/V = \int_0^\Lambda d^d\mathbf{q}/(2\pi)^d$.) Finally, a renormalization of K is obtained from

$$
\begin{aligned}
\langle \mathcal{U}_1^c\rangle_0^> =& -\frac{K}{2}\int \frac{d^d\mathbf{q}_1\, d^d\mathbf{q}_2\, d^d\mathbf{q}_3}{(2\pi)^{3d}}(\mathbf{q}_1\cdot\mathbf{q}_3) \\
&\left\langle \pi_\alpha^>(\mathbf{q}_2)\pi_\beta^>(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3)\right\rangle_0^> \pi_\alpha^<(\mathbf{q}_1)\pi_\beta^<(\mathbf{q}_3) \\
=&\frac{K}{2}\int_0^{\Lambda/b}\frac{d^d\mathbf{q}}{(2\pi)^d}\, q^2|\vec{\pi}^{\,<}(\mathbf{q})|^2 \times \frac{I_d(b)}{K},
\end{aligned}
\tag{VII.17}
$$

where

$$
I_d(b) \equiv \int_{\Lambda/b}^{\Lambda}\frac{d^d\mathbf{k}}{(2\pi)^d}\,\frac{1}{k^2} = \frac{K_d\Lambda^{d-2}\left(1-b^{2-d}\right)}{(d-2)}.
\tag{VII.18}
$$

The coarse–grained Hamiltonian in eq.(VII.14) now equals

$$
\begin{aligned}
\beta\tilde{\mathcal{H}}\left[\vec{\pi}^{\,<}\right] =& V\delta f_b^0 + V\delta f_b^1 + \frac{K}{2}\left(1+\frac{I_d(b)}{K}\right)\int_0^{\Lambda/b}\frac{d^d\mathbf{q}}{(2\pi)^d}\, q^2|\vec{\pi}^{\,<}(\mathbf{q})|^2 \\
&+\frac{K}{2}\int \frac{d^d\mathbf{q}_1\, d^d\mathbf{q}_2\, d^d\mathbf{q}_3}{(2\pi)^{3d}}\pi_\alpha^<(\mathbf{q}_1)\,\pi_\alpha^<(\mathbf{q}_2)\,\pi_\beta^<(\mathbf{q}_3)\,\pi_\beta^<(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3)\,(\mathbf{q}_1\cdot\mathbf{q}_3) \\
&-\frac{\rho}{2}\int_0^{\Lambda/b}\frac{d^d\mathbf{q}}{(2\pi)^d}|\vec{\pi}^{\,<}(\mathbf{q})|^2 \times \left[1-\left(1-b^{-d}\right)\right] + \mathcal{O}(T^2).
\end{aligned}
\tag{VII.19}
$$

The most important consequence of coarse graining is the change of the elastic coefficient K to

$$
\tilde{K} = K\left(1+\frac{I_d(b)}{K}\right).
\tag{VII.20}
$$

After rescaling, $\mathbf{x}' = \mathbf{x}/b$, and renormalizing, $\vec{\pi}\,'(\mathbf{x}) = \vec{\pi}^{\,<}(\mathbf{x})/\zeta$, we obtain the renormalized Hamiltonian in real space as

$$\begin{aligned}-\beta\mathcal{H}' = &- V\delta f_b^0 - V\delta f_b^1 - \frac{\tilde{K}b^{d-2}\zeta^2}{2}\int d^d\mathbf{x}'(\nabla'\pi')^2 \\ &-\frac{Kb^{d-2}\zeta^4}{2}\int d^d\mathbf{x}'\left(\vec{\pi}\,'(\mathbf{x}')\nabla\vec{\pi}\,'(\mathbf{x}')\right)^2 + \frac{\rho\zeta^2}{2}\int d^d\mathbf{x}'\,\pi'(\mathbf{x}')^2 + \mathcal{O}(T^2).\end{aligned} \tag{VII.21}$$

The easiest method for obtaining the rescaling factor ζ, is to take advantage of the rotational symmetry of spins. After averaging over the short wavelength modes, the spin is

$$\begin{aligned}\left\langle\tilde{\vec{s}}\right\rangle_0^> &= \left\langle\left(\pi_1^< + \pi_1^>, \cdots, \sqrt{1-(\vec{\pi}^{\,<}+\vec{\pi}^{\,>})^2}\right)\right\rangle_0^> \\ &= \left(\pi_1^<, \cdots, 1 - \frac{(\vec{\pi}^{\,<})^2}{2} - \left\langle\frac{(\vec{\pi}^{\,>})^2}{2}\right\rangle_0^> + \cdots\right) \\ &= \left(1 - \left\langle\frac{(\vec{\pi}^{\,>})^2}{2}\right\rangle_0^> + \mathcal{O}(T^2)\right)\left(\pi_1^<, \cdots, \sqrt{1-(\vec{\pi}^{\,<})^2}\right).\end{aligned} \tag{VII.22}$$

We thus identify

$$\zeta = 1 - \left\langle\frac{(\vec{\pi}^{\,>})^2}{2}\right\rangle_0^> + \mathcal{O}(T^2) = 1 - \frac{(n-1)}{2}\frac{I_d(b)}{K} + \mathcal{O}(T^2), \tag{VII.23}$$

as the length of the coarse-grained spin. The renormalized coupling constant in eq.(VII.21) is now obtained from

$$\begin{aligned}K' &= b^{d-2}\zeta^2\tilde{K} \\ &= b^{d-2}\left[1 - \frac{n-1}{2K}I_d(b)\right]^2 K\left[1 + \frac{1}{K}I_d(b)\right] \\ &= b^{d-2}K\left[1 - \frac{n-2}{K}I_d(b) + \mathcal{O}\left(\frac{1}{K^2}\right)\right].\end{aligned} \tag{VII.24}$$

For infinitesimal rescaling, $b = (1+\delta\ell)$, the shell integral results in

$$I_d(b) = K_d\Lambda^{d-2}\delta\ell. \tag{VII.25}$$

The differential recursion relation corresponding to eq.(VII.24) is thus

$$\frac{dK}{d\ell} = (d-2)K - (n-2)K_d\Lambda^{d-2}. \tag{VII.26}$$

Alternatively, the scaling of temperature $T = K^{-1}$, is

$$\frac{dT}{d\ell} = -\frac{1}{K^2}\frac{dK}{d\ell} = -(d-2)T + (n-2)K_d\Lambda^{d-2}T^2. \tag{VII.27}$$

It may appear that we should also keep track of the evolution of the coefficients of the two terms in $\mathcal{U}_1$ under RG. In fact, spherical symmetry ensures that the coefficient of the quartic term is precisely the same as K at all orders. The apparent difference between the two is of order of $\mathcal{O}(T^2)$, and will vanish when all terms at this order are included. The coefficient of the second order term in $\mathcal{U}_1$ merely tracks the density of points and also has trivial renormalization.

The behavior of temperature under RG changes drastically at $d = 2$. For $d < 2$, the linear flow is away from zero, indicating that the ordered phase is unstable and there is no broken symmetry. For $d > 2$, small T flows back to zero, indicating that the ordered phase is stable. The flows for $d = 2$ are controlled by the second ordered term which changes sign at $n = 2$. For $n > 2$ the flow is towards high temperatures, indicating that Heisenberg and higher spin models are disordered. The situation for $n = 2$ is ambiguous, and it can in fact be shown that $dT/d\ell$ is zero to all orders. This special case will be discussed in more detail in the next section. For $d > 2$ and $n > 2$, there is a phase transition at the fixed point,

$$T^* = \frac{\epsilon}{(n-2)K_d\Lambda^{d-2}} = \frac{2\pi\epsilon}{(n-2)} + \mathcal{O}(\epsilon^2), \tag{VII.28}$$

where $\epsilon = d - 2$ is used as a small parameter. The recursion relation at order of ϵ is

$$\frac{dT}{d\ell} = -\epsilon T + \frac{(n-2)}{2\pi}T^2. \tag{VII.29}$$

Stability of the fixed point is determined by the linearized recursion relation

$$\left.\frac{d\delta T}{d\ell}\right|_{T^*} = \left[-\epsilon + \frac{(n-2)}{\pi}T^*\right]\delta T = \left[-\epsilon + 2\epsilon\right]\delta T = \epsilon\delta T, \qquad \Longrightarrow \qquad y_t = \epsilon \quad . \tag{VII.30}$$

The thermal eigenvalue, and the resulting exponents $\nu = 1/\epsilon$, and $\alpha = 2-(2+\epsilon)/\epsilon \approx -2/\epsilon$, are independent of n at this order.

The magnetic eigenvalue can be obtained by adding a term $-\vec{h}\cdot\int d^d\mathbf{x}\,\vec{s}(\mathbf{x})$, to the Hamiltonian. Under the action of RG, $h' = b^d\zeta h \equiv b^{y_h}h$, with

$$b^{y_h} = b^d\left[1 - \frac{n-1}{2K}I_d(b)\right]. \tag{VII.31}$$

For an infinitesimal rescaling

$$1 + y_h \delta\ell = (1 + d\delta\ell)\left(1 - \frac{n-1}{2} T^* K_d \Lambda^{d-2} \delta\ell\right), \tag{VII.32}$$

leading to

$$y_h = d - \frac{n-1}{2(n-2)}\epsilon = 1 + \frac{n-3}{2(n-2)}\epsilon + \mathcal{O}(\epsilon^2), \tag{VII.33}$$

which does depend on n. Using exponent identities, we find

$$\eta = 2 + d - 2y_h = \frac{\epsilon}{n-2}. \tag{VII.34}$$

The exponent η is zero at the lowest order in a $4-d$ expansion, but appears at first order in the vicinity of two dimensions. The actual values of the exponents calculated at this order are not very satisfactory.