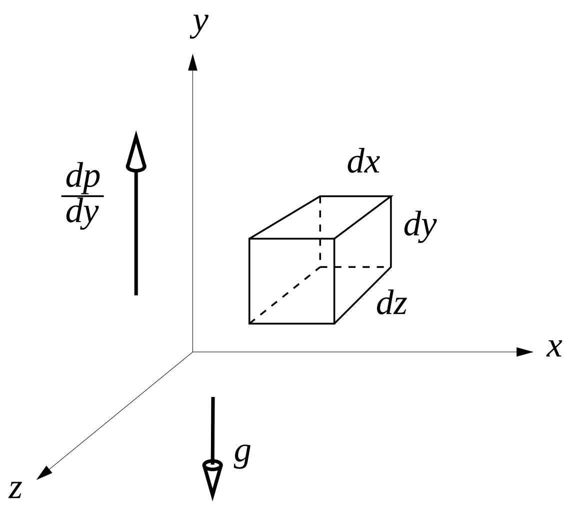
Fluids – Lecture 2 Notes

1. Hydrostatic Equation
2. Manometer
3. Buoyancy Force

Reading: Anderson 1.9

Hydrostatic Equation

Consider a fluid element in a pressure gradient in the vertical y direction. Gravity is also present.



If the fluid element is at rest, the net force on it must be zero. For the vertical y-force in particular, we have

$$\begin{aligned}
\text{Pressure force} \;+\; \text{Gravity force} \;&=\; 0 \\
p\,dA \;-\; \left(p + \frac{dp}{dy}dy\right) dA \;-\; \rho\, g\, d\mathcal{V} \;&=\; 0 \\
-\frac{dp}{dy}\,dy\,dA \;-\; \rho\, g\, d\mathcal{V} \;&=\; 0
\end{aligned}$$

The area on which the pressures act is $dA = dx\,dz$, and the volume is $d\mathcal{V} = dx\,dy\,dz$, so that

$$\begin{aligned}
-\frac{dp}{dy}\,dx\,dy\,dz \;-\; \rho\, g\, dx\,dy\,dz \;&=\; 0 \\
dp \;&=\; -\rho g\, dy \qquad (1)
\end{aligned}$$

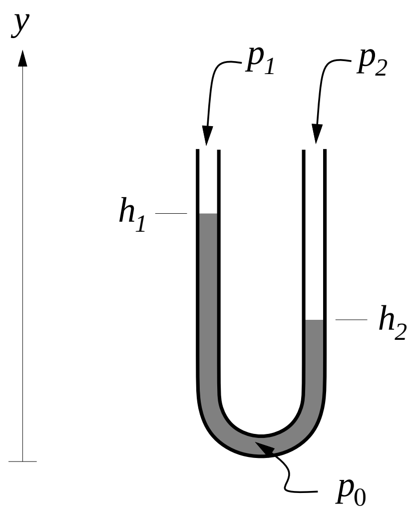
which is the differential form of the *Hydrostatic Equation* If we make the further assumption that the density is constant, this equation can be integrated to the equivalent integral form.

$$p(y) \;=\; p_0 \;-\; \rho g y \qquad (2)$$

The constant of integration p_0 is the pressure at the particular location $y = 0$. Note that this integral form is valid *provided the density is constant within the region of interest*

Application to a Manometer

A manometer is a U-shaped tube partially filled with a liquid, as shown in the figure. Two different pressures p_1 and p_2 are applied to the two legs of the tube, causing the two liquid columns to have different heights h_1 and h_2.



We now pick p_0 to be the pressure at some point of the tube (at the bottom for instance), and apply equation (2) to each leg of the tube.

$$
\begin{aligned}
p_1 &= p_0 - \rho g h_1 \\
p_2 &= p_0 - \rho g h_2
\end{aligned}
$$

Subtracting these two equations then gives the difference of the pressures in terms of the liquid height difference.

$$
p_2 - p_1 = \rho g (h_1 - h_2) \tag{3}
$$

If tube 1 is left open to the atmosphere, so that $p_1 = p_{\mathrm{atm}}$, then p_2 can be measured simply by applying it to tube 2, measuring the height difference $\Delta h = h_1 - h_2$, and applying equation (3) above.

$$
p_2 = p_{\mathrm{atm}} + \rho g \, \Delta h
$$

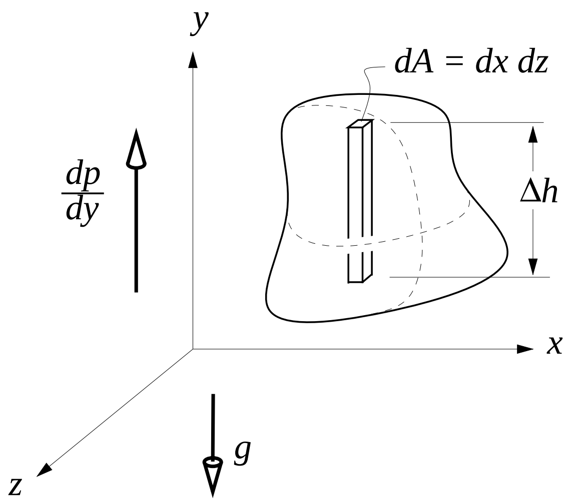
This requires knowing the density ρ of the fluid to sufficient accuracy.

Buoyancy

Now consider an object of arbitrary shape immersed in the pressure gradient. The object's volume can be divided into vertical "matchstick" volumes, each of infinitesimal cross-sectional area $dA = dx\,dz$, and finite height Δh.

The vertical y-direction pressure force on each volume is

$$
\begin{aligned}
dF &= p\,dA - \left(p + \frac{dp}{dy}\Delta h\right) dA \\
dF &= -\frac{dp}{dy}\,\Delta h\,dA \\
dF &= \rho g\,d\mathcal{V}
\end{aligned}
$$



where dp/dy has been replaced by $-\rho g$ using the Hydrostatic Equation (1), and the volume of the infinitesimal volume is $\Delta h \, dA = d\mathcal{V}$. Integrating the last equation above then gives the total buoyancy force on the object.

$$F = \rho g \mathcal{V}$$

It is important to note that $\mathcal{V}$ is the overall volume of the object, while ρ is the density of the fluid. The product $\rho\mathcal{V}$ is recognized as the mass of the fluid displaced by the object, and $\rho g\mathcal{V}$ is the corresponding weight, giving the well known *Archimedes Principle*:

$$\text{Buoyancy force on body} = \text{Weight of fluid displaced by body}$$