Notes on the Control of an Aircraft with Throttle Only

There are a few (rare) examples of flight control system failures that left aircraft with no elevator control. In these cases, pilots have attempted to control the vertical position of the aircraft using thrust changes only. This turns out to be very hard to do, because the resulting dynamics are hard for a pilot to control. Below, we derive the dynamics of an aircraft under those conditions. To simplify the problem, we use the *phugoid approximation*.

In the phugoid approximation, the aircraft is treated as a point mass, and the pitching dynamics of the aircraft are ignored. Furthermore, only the motion of the aircraft in the vertical plane (the longitudinal motions) are considered. The states of the aircraft are then

$$\begin{aligned} h &= \text{altitude of aircraft} \\ v &= \text{velocity of aircraft} \\ \gamma &= \text{flight path angle of aircraft} \end{aligned} \tag{1}$$

The flight path angle of the aircraft is the angle of the velocity vector with respect to horizontal, with a positive γ indicating that the velocity vector is above the horizon.

The dynamics of the aircraft can be fairly easily derived, using only Newton's laws, and a little aerodynamics. First, find $\dot{h}$, the rate of change of the altitude. $\dot{h}$ is determined completely by kinematics, so that

$$\dot{h} = v \sin\gamma \tag{2}$$

Next, $\dot{v}$ is the rate of change of the velocity vector, which is the acceleration of the aircraft in the direction of the velocity vector, so that

$$\dot{v} = \frac{T - D}{m} - \frac{mg\sin\gamma}{m} = \frac{T - D}{m} - g\sin\gamma \tag{3}$$

where T is the thrust, D is the drag, m is the mass of the airplane, and g is the acceleration due to gravity. Finally, we must find $\dot{\gamma}$, the rate of change of the flight path angle. This one is a little tricky, because we have essentially defined a non-inertial coordinate frame for the aircraft. It turns out that $\dot{\gamma}$ is the acceleration perpendicular to the flight path, divided by the aircraft velocity, so that

$$\dot{\gamma} = \frac{L}{mv} - \frac{mg\cos\gamma}{mv} = \frac{L}{mv} - \frac{g\cos\gamma}{v} \tag{4}$$

To complete the modeling, we need to model the aerodynamics of the aircraft. In the phugoid approximation, it is assumed that the coefficients of lift and drag (C_L and C_D) are constant. Then Equation (3) becomes

$$\dot{v} = \frac{T - \frac{1}{2}\rho v^2 S C_D}{m} - g \sin\gamma \tag{5}$$

where S is the reference wing area of the airplane. We can simplify this last equation a bit by defining the drag at the nominal flight condition as

$$D_0 = \frac{1}{2}\rho v_0^2 S C_D \tag{6}$$

Then Equation (5) becomes

$$\dot{v} = \frac{T}{m} - \frac{D_0}{m}\frac{v^2}{v_0^2} - g \sin\gamma \tag{7}$$

Similarly, Equation (4) becomes

$$\dot{\gamma} = \frac{L_0}{mv}\frac{v^2}{v_0^2} - \frac{g\cos\gamma}{v} = \frac{L_0}{m}\frac{v}{v_0^2} - \frac{g\cos\gamma}{v} \tag{8}$$

But in equilibrium flight, the nominal lift L_0 must equal the weight of the aircraft, mg. Therefore,

$$\dot{\gamma} = g\frac{v}{v_0^2} - \frac{g\cos\gamma}{v} \tag{9}$$

Collecting these together, we have the nonlinear equations that describe the motion of the aircraft:

$$\dot{h} = v \sin\gamma \tag{10}$$

$$\dot{v} = \frac{T}{m} - \frac{D_0}{m}\frac{v^2}{v_0^2} - g \sin\gamma \tag{11}$$

$$\dot{\gamma} = g\frac{v}{v_0^2} - \frac{g\cos\gamma}{v} \tag{12}$$

The next step is to linearize the equations of motion about the equilibrium flight condition. In equilibrium flight,

$$L_0 = mg \tag{13}$$

$$T_0 = D_0 = \frac{L_0}{(L_o/D_0)} \tag{14}$$

$$\gamma_0 = 0 \tag{15}$$

We consider small perturbations about the nominal, so that, for example, $v = v_0 + \delta v$, where δv is the perturbation in velocity. In general, a nonlinear function $f(x)$ can be linearized about a nominal point x_0 by using the Taylor series expansion

$$f(x) = f(x_0 + \delta x) \approx f(x_0) + f'(x_0)\delta x \tag{16}$$

if the perturbations are small. For our equation, we obtain

$$\dot{h} = \delta\dot{h} = v \sin\gamma \approx v_0\, \delta\gamma \tag{17}$$

$$\dot{v} = \delta\dot{v} = \frac{T_0 + \delta T}{m} - \frac{D_0}{m}\frac{(v_0 + \delta v)^2}{v_0^2} - g \sin\gamma \approx \frac{1}{m}\delta T - \frac{2D_0}{mv_0}\delta v - g\,\delta\gamma \tag{18}$$

$$\dot{\gamma} = \delta\dot{\gamma} = g\frac{v_0 + \delta v}{v_0^2} - \frac{g\cos\gamma}{v_0 + \delta v} \approx \left(\frac{2g}{v_0^2}\right)\delta v \tag{19}$$

The equations above can be expressed in matrix form as

$$\begin{bmatrix} \delta\dot{h} \\ \delta\dot{v} \\ \delta\dot{\gamma} \end{bmatrix} = \begin{bmatrix} 0 & 0 & v_0 \\ 0 & -\frac{2gD_0}{v_0 L_0} & -g \\ 0 & \frac{2g}{v_0^2} & 0 \end{bmatrix} \begin{bmatrix} \delta h \\ \delta v \\ \delta\dot{\gamma} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \\ 0 \end{bmatrix} \delta T \tag{20}$$

The equation above can be Laplace transformed to obtain

$$s\begin{bmatrix} H(s) \\ V(s) \\ \Gamma(s) \end{bmatrix} = \begin{bmatrix} 0 & 0 & v_0 \\ 0 & -\frac{2gD_0}{v_0 L_0} & -g \\ 0 & \frac{2g}{v_0^2} & 0 \end{bmatrix} \begin{bmatrix} H(s) \\ V(s) \\ \Gamma(s) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \\ 0 \end{bmatrix} T(s) \tag{21}$$

where we have ignored the initial condition terms, since they should all be zero. Solving for $H(s)$ in terms of $T(s)$ yields

$$H(s) = \frac{2g}{mv_0}\,\frac{1}{s\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)}\,T(s) \tag{22}$$

where the *natural frequency* of the phugoid mode is

$$\omega_n = \sqrt{2}\frac{g}{v_0} \tag{23}$$

and the *damping ratio* is

$$\zeta = \frac{1}{\sqrt{2}(L_0/D_0)} \tag{24}$$

Consider an aircraft cruising at 250 knots (about 129 m/s), with a lift-to-drag ratio of 15. Then the natural frequency is

$$\omega_n = \sqrt{2}\frac{g}{v_0} = 0.108 \text{ rad/s} = 0.0171 \text{ Hz} \tag{25}$$

and the damping ratio is

$$\zeta = \frac{1}{\sqrt{2}(L_0/D_0)} = 0.047 \tag{26}$$

Therefore, the transfer function of the system is

$$G(s) = \frac{2g}{mv_0}\frac{1}{s\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)} \tag{27}$$

Suppose the pilot attempts to increase the altitude a little by increasing the throttle momentarily, and then returning the throttle back to the nominal position. The response to this input is essentially the impulse response, if the time of the input is very short compared to the response time of the system. The impulse response of the system (normalized by the factor of $\frac{2g}{mv_0}$), is shown in Figure 1. Note that the application of throttle causes a lightly-damped oscillation, called the *phugoid mode*. In this case, the period of the phugoid mode is about 1 minute. The period of the phugoid mode depends only on the velocity of the aircraft, so every airplane flying at 250 knots will have a similar response. So the pilot does succeed in changing the altitude of the aiorplane is stable flight, but in the process, induces an oscillation that persists for nearly 10 minutes.

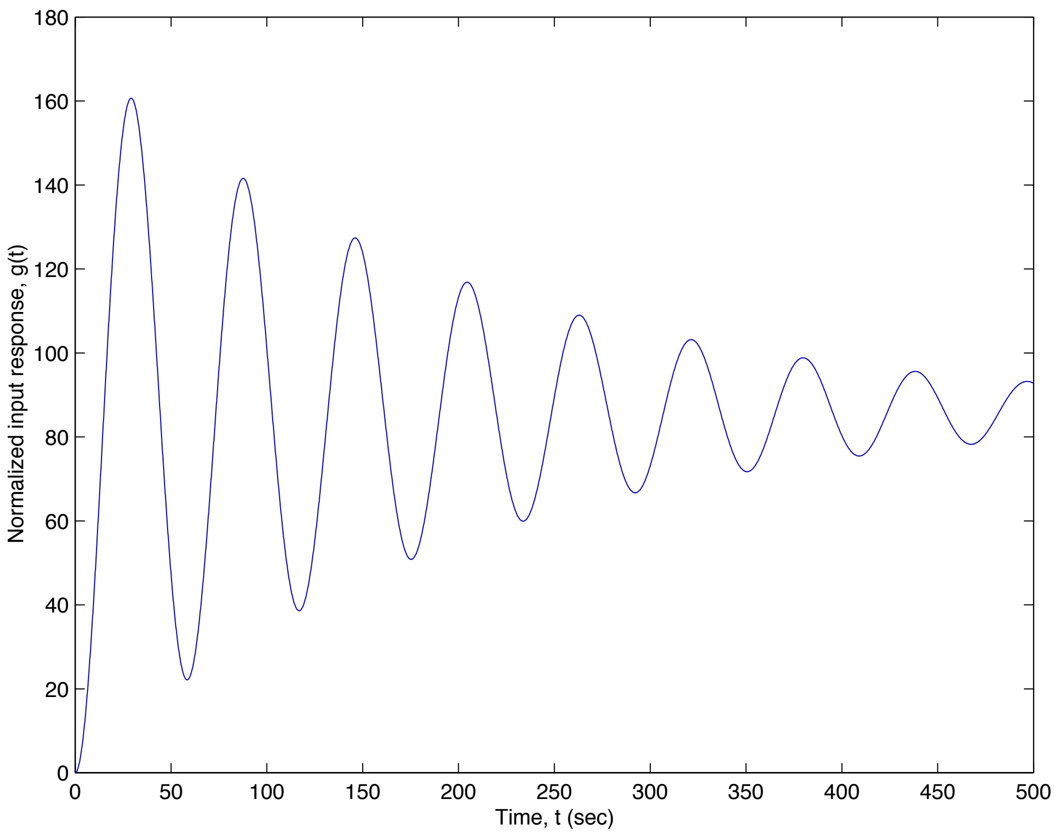

Figure 1: Impulse response of aircraft.