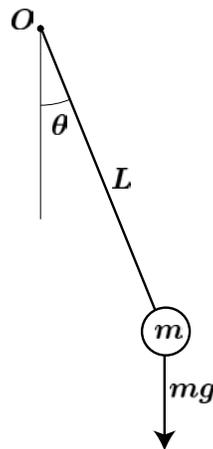


## Lecture L24 - Pendulums

A pendulum is a rigid body suspended from a fixed point (hinge) which is offset with respect to the body's center of mass. If all the mass is assumed to be concentrated at a point, we obtain the idealized *simple pendulum*. Pendulums have played an important role in the history of dynamics. Galileo identified the pendulum as the first example of synchronous motion, which led to the first successful clock developed by Huygens. This clock incorporated a feedback mechanism that injected energy into the oscillations (the escapement, a mechanism used in timepieces to control movement and to provide periodic energy impulses to a pendulum or balance) to compensate for friction losses. In addition to horology (the science of measuring time), pendulums have important applications in gravimetry (the measurement of the specific gravity) and inertial navigation.

### Simple Pendulum

Consider a simple pendulum of mass  $m$  and length  $L$ .



The equation of motion can be derived from the conservation of angular momentum about the hinge point,  $O$ ,

$$I_O \ddot{\theta} = -mgL \sin \theta .$$

Since the moment of inertia is simply  $I_O = mL^2$ , we obtain the following non-linear equation of motion,

$$\ddot{\theta} + \frac{g}{L} \sin \theta = 0 . \tag{1}$$

Multiplying this equation by  $\dot{\theta}$ , we can write,

$$\frac{d}{dt} \left( \frac{1}{2} \dot{\theta}^2 - \frac{g}{L} \cos \theta \right) = 0,$$

which implies that  $\dot{\theta}^2 - (2g/L) \cos \theta = \text{constant}$ . Setting  $\theta = \theta_{max}$ , when  $\dot{\theta} = 0$  we have,

$$\dot{\theta} = \pm \sqrt{\frac{2g}{L} \sqrt{(\cos \theta - \cos \theta_{max})}} .$$

This equation cannot be integrated further in an explicit manner. Its solution must be expressed in terms of, so called, elliptic functions. The period of the oscillation,  $T$ , is obtained by multiplying by four the time it takes for the pendulum to go from  $\theta = 0$  to  $\theta = \theta_{max}$ . Thus,

$$T = \frac{4}{\sqrt{(2g/L)}} \int_0^{\theta_{max}} \frac{d\theta}{\sqrt{\cos \theta - \cos \theta_{max}}} .$$

Again, this is an integral which cannot be evaluated explicitly, but can be approximated, assuming that  $\theta_{max}$  is not very large, as (the algebra is omitted here),

$$T \approx 2\pi \sqrt{\frac{L}{g}} \left( 1 + \frac{\theta_{max}^2}{16} \right) . \quad (2)$$

### Small amplitude approximation

If we assume that the amplitude of pendulum's oscillation is small, then  $\sin \theta \approx \theta$ , and the equation of motion, given by 1, becomes linear,

$$\ddot{\theta} + \frac{g}{L} \theta = 0 . \quad (3)$$

This expression is much simpler than equation 1, and has solutions of the form,

$$\theta = A \sin \omega_n t + B \cos \omega_n t ,$$

where  $\omega_n = \sqrt{g/L}$  is the natural frequency of oscillation. It is clear that these solutions are periodic, and the period is given by

$$T = \frac{2\pi}{\omega_n} = 2\pi \sqrt{\frac{L}{g}} . \quad (4)$$

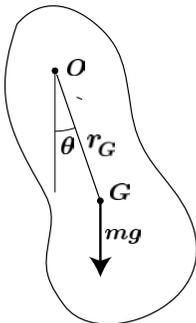
Setting  $\theta = 0$  for  $t = 0$ , and  $\theta = \theta_{max}$  for  $t = T/4$ , we obtain,

$$\theta = \theta_{max} \sin \omega_n t .$$

We observe that, in the small amplitude case, the period is independent of  $\theta_{max}$ . This is called *synchronism* and is central to time-keeping functions in clocks. This means that, provided the amplitude is small, small changes in amplitude due to friction or other disturbances have little effect on the period. Comparing expression 4 with the approximate solution of the non-linear problem given by equation 2, we see that for amplitudes of, say,  $\theta_{max} = 30^\circ$ , the difference between the two expressions is only about 1.7%.

## Compound Pendulum

In the case of a compound pendulum, we can proceed in a similar manner.



Conservation of angular momentum about  $O$  gives,

$$I_O \ddot{\theta} = -mgr_G \sin \theta .$$

Expressing  $I_O$  in terms of the radius of gyration,  $k_O$ ,

$$I_O = mk_O^2$$

we have,

$$\ddot{\theta} + \frac{gr_G}{k_O^2} \sin \theta = 0 . \quad (5)$$

We note that this equation is the same as equation 1 for the simple pendulum, if we identify the term  $g/L$  in equation 1 with the term  $gr_G/k_O^2$  in equation 5. This leads to the definition of an equivalent length,  $L_{equiv}$ , as,

$$L_{equiv} = \frac{k_O^2}{r_G} = \frac{k_G^2 + r_G^2}{r_G} . \quad (6)$$

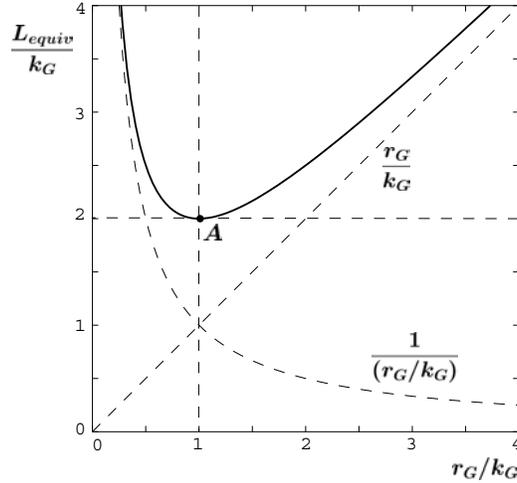
Here we have used the fact that  $I_O = I_G + mr_G^2$ , and therefore  $k_O^2 = k_G^2 + r_G^2$ . Thus, we have that the motion of a compound pendulum is identical to that of a simple pendulum of equivalent length  $L_{equiv}$ , given by equation 6. Using the small amplitude approximation, the period of the compound pendulum will be

$$T = 2\pi \sqrt{\frac{L_{equiv}}{g}} .$$

One question we may want to ask is whether, for a given body ( $k_G$  fixed), we can make the period (or  $L_{equiv}$ ) arbitrarily small by choosing  $r_G$  (or the hinge point  $O$ ) appropriately. From equation 6, we can write

$$\frac{L_{equiv}}{k_G} = \frac{r_G}{k_G} + \frac{1}{(r_G/k_G)} ,$$

which shows that, when we try to reduce  $L_{equiv}/k_G$  by reducing  $r_G$ , the term  $r_G/k_G$  is reduced, but the term  $1/(r_G/k_G)$  increases. This situation is shown in the graph, which also shows that the minimum value for  $L_{equiv}/k_G$  is 2, a value which is attained for  $r_G = k_G$ .



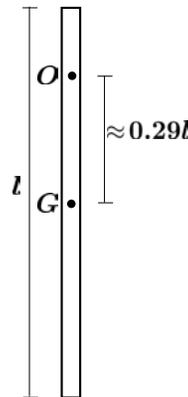
In conclusion, we have that for a given solid, the shortest equivalent length, and hence the fastest pendulum, occurs when it is suspended from a point which is at a distance  $r_G = k_G$  from the center of mass. In this case, the minimum equivalent length is

$$(L_{equiv})_{min} = 2k_G .$$

**Example**

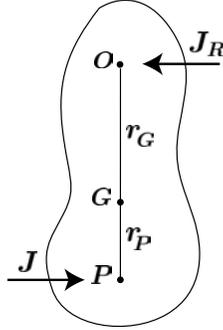
**Minimum period pendulum**

Consider a uniform rod of length  $l$ . In this case,  $I_G = ml^2/12$ , and  $k_G = l/\sqrt{12} \approx 0.29l$ . Therefore, the fastest pendulum is obtained when the bar is suspended from a distance  $0.29l$  away from the center of mass. We note that if the suspension point is moved slightly, the period of the pendulum will increase. However, moving the suspension point will make practically no difference in the frequency, because the tangent at point A (see graph) is horizontal. This fact has been used in the construction of extremely accurate pendulums for clocks.



## Center of Percussion

The center of percussion associated with a given center of motion was introduced in lecture L24. In the case of a pendulum, the desired center of motion is the hinge,  $O$ . The center of percussion is on the extension of the line that connects  $O$  with the center of mass.



From lecture L24, we know that the distance,  $r_P$ , between the center of percussion,  $P$ , and the center of mass,  $G$ , is given by,

$$r_P = \frac{k_G^2}{r_G} . \quad (7)$$

Recall that the center of percussion is the point at which we should strike the pendulum with a horizontal impulse so that  $O$  becomes the instantaneous center of motion. It is clear that, if  $O$  is the instantaneous center of motion, the horizontal reaction needed to keep the pendulum in place will be zero and, as a consequence, the impulse will not be felt at  $O$ . In fact, this can be verified directly in a straightforward manner. Let us assume that the pendulum is given an impulse  $J$  at a distance  $r_P$  below the center of mass,  $G$ . Conservation of momentum in the horizontal direction and conservation of angular momentum around  $G$  imply that,

$$\begin{aligned} mv_G &= J - J_R \\ I_G \omega &= Jr_P + J_R r_G . \end{aligned}$$

Solving for  $J_R$ , we have that,

$$J_R = J \frac{I_G - mr_G r_P}{I_G + mr_G^2} = J \frac{k_G^2 - r_G r_P}{I_O} ,$$

which clearly shows that the impulse reaction,  $J_R$ , is zero when  $r_P$  satisfies equation 7.

We note that, for a given body,  $r_P$  will be large when  $r_G$  is very small, and, as a consequence, the center of percussion will not be a material point in the body.

## Reversibility

It turns out that if the pendulum is suspended from point  $P$ , the roles of  $r_G$  and  $r_P$  are reversed. In particular, point  $O$  is the center of percussion relative to the instantaneous center of motion,  $P$ . Using

equations 6 and 7, the equivalent length of a compound pendulum can be written as,

$$L_{equiv} = \frac{k_G^2 + r_G^2}{r_G} = \frac{r_P r_G + r_G^2}{r_G} = r_G + r_P ,$$

and the corresponding period is,

$$T = 2\pi \sqrt{\frac{L_{equiv}}{g}} = 2\pi \sqrt{\frac{r_G + r_P}{g}} .$$

It should be clear that when the pendulum is suspended from point  $P$ , the equivalent length does not change, since,

$$\frac{k_G^2 + r_P^2}{r_P} = \frac{r_P r_G + r_P^2}{r_P} = r_G + r_P \equiv L_{equiv} .$$

### The Kater Pendulum

The reversibility of point  $O$  and  $P$  described above is the principle of the reversible pendulum, invented by Kater to measure gravity with high accuracy. Kater's pendulum consists of a long bar, equipped with two fixed knife edges at an accurately known distance  $L$ , and with some moveable masses positioned along the bar. The positions of these masses on the bar are adjusted until the periods associated with suspension from either knife edge are precisely equal to, say,  $T$ . This guarantees that each point is the center of percussion relative to the other, and, thus,  $L = L_{equiv}$ . The local gravity is then given by,

$$g = L_{equiv} \left( \frac{2\pi}{T} \right)^2 .$$

### The Schuler Pendulum

Consider a pendulum suspended in vertical position from point  $O$ . If the support is suddenly accelerated in the horizontal direction, the pendulum will rotate due to the inertial forces acting on its center of mass. It is not difficult to show that the instantaneous center of motion in this case will be precisely the center of percussion associated with  $O$ .

Now, imagine a pendulum on the earth's surface whose center of percussion is at the earth's center. When the pendulum's support is accelerated, the pendulum rotates in a way such that it always points towards the center of the earth. Such a pendulum always keeps itself vertical regardless of the acceleration, and would be of obvious interest for applications such as inertial navigation.

It is interesting to see that, although conceptually correct, the construction of such a pendulum would pose serious technological challenges. If we assume an earth radius of  $R = 6370$  km, then  $r_P \approx R$ . Considering a pendulum with, say,  $k_G = 0.3$ m, the distance between the support and the center of mass would have to be,

$$r_G = \frac{0.3^2}{6.37 \times 10^6} = 1.5 \times 10^{-8} \text{m} = 0.015 \mu\text{m} .$$

This distance approaches atomic dimensions and, in fact, such a pendulum has never been constructed using purely mechanical means. It is, however, routinely realized by replacing the physical pendulum with a combination of gyroscope and accelerometers having the same dynamics.

It is interesting to note that the period of the Schuler pendulum is given by

$$T_{Schuler} = 2\pi\sqrt{\frac{L_{equiv}}{g}} \approx 2\pi\sqrt{\frac{R}{g}} \approx 84.4\text{min.} ,$$

which is exactly the same period as that of a circular orbit around the earth at zero altitude.

## References

- [1] M. Martinez-Sanchez, *Unified Engineering Notes*, Course 95-96.
- [2] J. P. Den Hartog, *Mechanics*, Dover, 1961

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