Linearized Subsonic Flow

We desire to solve

$$(1-M_\infty^2)\phi_{xx}+\phi_{yy}+\phi_{zz}=0 \tag{1.1}$$

$$0<M_\infty<1 \tag{1.2}$$

$$(1-M_\infty^2)\geq 0 \tag{1.3}$$

In two dimensions, we have

$$(1-M_\infty^2)\phi_{xx}+\phi_{yy}=0 \tag{1.4}$$

$$v'=U_\infty\left(\frac{\partial y}{\partial x}\right)_{\text{BODY}} \tag{1.5}$$

$$u'\rightarrow 0, \quad x,y\rightarrow\infty \tag{1.6}$$

$$v'\rightarrow 0, \quad x,y\rightarrow\infty \tag{1.7}$$

Now let

$$\beta=\sqrt{1-M_\infty^2} \tag{1.8}$$

and transform independent coordinates as follows:

$$\xi=x \tag{1.9}$$

$$\eta=\beta y \tag{1.10}$$

And likewise, the dependent perturbation velocity potential

$$\tilde{\phi}(\xi,\eta)=\beta\phi(x,y) \tag{1.11}$$

This series of transformations lead to the following

$$\frac{\partial\xi}{\partial x}=1 \quad \frac{\partial\xi}{\partial y}=0 \quad \frac{\partial\eta}{\partial x}=0 \quad \frac{\partial\eta}{\partial y}=\beta \tag{1.12}$$

$$\phi_x=\frac{\partial\phi}{\partial x}=\frac{1}{\beta}\frac{\partial\tilde{\phi}}{\partial x}=\frac{1}{\beta}\left[\frac{\partial\tilde{\phi}}{\partial\xi}\frac{\partial\xi}{\partial x}+\frac{\partial\tilde{\phi}}{\partial\eta}\frac{\partial\eta}{\partial x}\right]=\frac{1}{\beta}\frac{\partial\tilde{\phi}}{\partial\xi}=\frac{1}{\beta}\tilde{\phi}_\xi \tag{1.13}$$

$$\phi_{xx}=\frac{1}{\beta}\tilde{\phi}_{\xi\xi} \tag{1.14}$$

$$\phi_y = \frac{\partial \phi}{\partial y} = \frac{1}{\beta}\frac{\partial \widetilde{\phi}}{\partial y} = \frac{1}{\beta}\left[\frac{\partial \widetilde{\phi}}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial \widetilde{\phi}}{\partial \eta}\frac{\partial \eta}{\partial y}\right] = \frac{\partial \widetilde{\phi}}{\partial \eta} = \widetilde{\phi}_\eta \tag{1.15}$$

$$\phi_{yy} = \beta \widetilde{\phi}_{\eta\eta} \tag{1.16}$$

Our transformed governing equation becomes

$$\beta^2\left(\frac{1}{\beta}\widetilde{\phi}_{\xi\xi}\right) + \beta\widetilde{\phi}_{\eta\eta} = 0 \tag{1.17}$$

or

$$\widetilde{\phi}_{\xi\xi} + \widetilde{\phi}_{\eta\eta} = 0 \tag{1.18}$$

Our analysis drives us to the following question:

HOW CAN WE EXPLOIT INCOMPRESSIBLE RESULTS TO ACCOUNT FOR COMPRESSIBILITY EFFECTS?

Compare the forms

$$\phi_{xx} + \phi_{yy} = 0, \quad M_\infty = 0 \tag{1.19}$$

$$\widetilde{\phi}_{\xi\xi} + \widetilde{\phi}_{\eta\eta} = 0, \quad \beta > 0 \tag{1.20}$$

Consider the boundary condition on the airfoil surface.

$$y = f(x), \quad \text{airfoil shape in } x, y \tag{1.21}$$

$$\eta = q(\xi), \quad \text{airfoil shape in } \xi, \eta \tag{1.22}$$

Our boundary condition may be expressed as follows:

$$U_\infty \frac{df}{dx} = \frac{\partial \phi}{\partial y} = \frac{1}{\beta}\frac{\partial \widetilde{\phi}}{\partial y} = \frac{\partial \widetilde{\phi}}{\partial \eta} \tag{1.23}$$

(x, y) space

Similarly in (ξ, η) space

$$U_\infty \frac{dq}{d\xi} = \frac{\partial \widetilde{\phi}}{\partial \eta} \tag{1.24}$$

(ξ, η) space

Therefore,

$$U_\infty \frac{df}{dx} = U_\infty \frac{dq}{d\xi} \tag{1.25}$$

or

$$\frac{df}{dx} = \frac{dq}{d\xi} \tag{1.26}$$

Conclusions

(a) The shape of the airfoil in x, y space is the same in ξ, η space.

(b) $\widetilde{\phi}, \xi, \eta$ implies that the compressible flow over an airfoil in x, y space is related to an incompressible flow in ξ, η space over the same airfoil.

Now, let's return to the pressure coefficient:

$$\begin{aligned} c_p &= -2\frac{u'}{U_\infty} \\ &= -2\frac{1}{U_\infty}\frac{\partial \phi}{\partial x} \\ &= -\frac{2}{U_\infty}\frac{1}{\beta}\frac{\partial \widetilde{\phi}}{\partial \xi} \end{aligned} \tag{1.27}$$

Let the incompressible pressure coefficient be

$$c_{p_0} \equiv -2\frac{\widetilde{u}^{'}}{U_\infty} = -2\frac{1}{U_\infty}\frac{\partial\widetilde{\phi}}{\partial\xi} \tag{1.28}$$

Therefore, substituting

$$c_p = \frac{1}{\beta}c_{p_0} \tag{1.29}$$

$$c_p = \frac{c_{p_0}}{\sqrt{1-M_\infty^2}} \tag{1.30}$$

This is the Prandtl-Glauert rule. It is a similarity rule that relates incompressible flow over a given two-dimensional profile to subsonic compressible flow over the same profile.
From above results, it can be shown that

$$c_L = \frac{c_{L_0}}{\sqrt{1-M_\infty^2}} \tag{1.31}$$

$$C_M = \frac{C_{M_0}}{\sqrt{1-M_\infty^2}} \tag{1.32}$$

$$u^{'} = \frac{\widetilde{u}}{\sqrt{1-M_\infty^2}} \tag{1.33}$$

What does it mean?