Unit 16
Bifurcation Buckling

Readings:

Rivello 14.1, 14.2, 14.4

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V. Stability and Buckling

Now consider the case of compressive loads and the *instability* they can cause. Consider only static instabilities (static loading as opposed to dynamic loading [e.g., flutter])

From Unified, defined instability via:

> "A system becomes unstable when a negative stiffness overcomes the natural stiffness of the structure."

(Physically, the more you push, it gives more and builds on itself)

Review some of the mathematical concepts. Limit initial discussions to columns.

Generally, there are two types of buckling/instability

- Bifurcation buckling
- Snap-through buckling

Bifurcation Buckling

There are two (or more) equilibrium solutions (thus the solution path "bifurcates")

from Unified…

Figure 16.1 **Representation of initially straight column under compressive load**

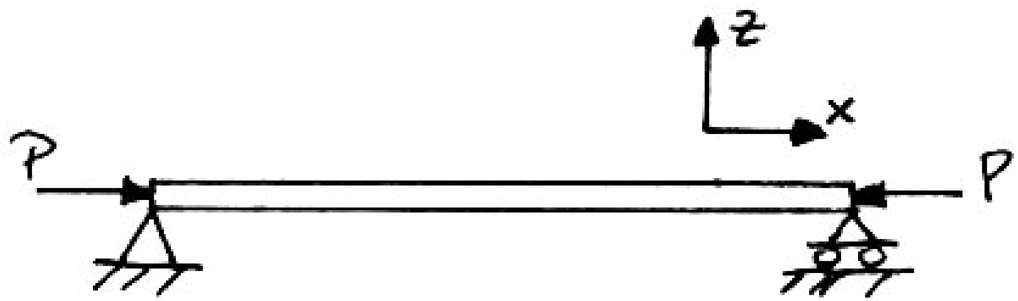
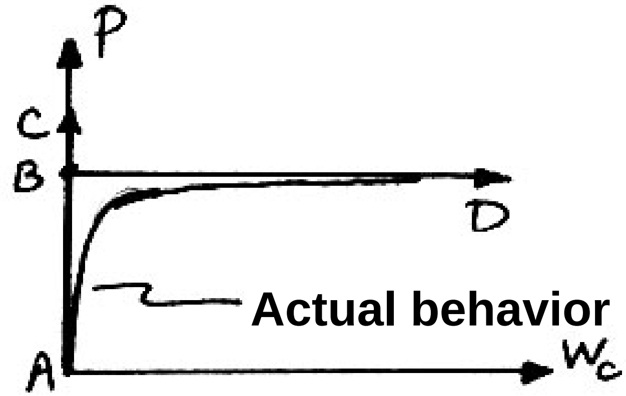


Figure 16.2 **Basic load-deflection behavior of initially straight column under compressive load**



<u>Note</u>: Bifurcation is a *mathematical* concept. The manifestations in an actual system are altered due to physical realities/imperfections. Sometimes these differences can be very important.

(first continue with ideal case…)

<u>Perfect behavior</u> } ABC - Equilibrium position, but unstable
BD - Equilibrium position

There are also other equilibrium positions

Imperfections cause the actual behavior to only follow this as asymptotes (will see later)

Snap-Though Buckling

Figure 16.3 **Representation of column with curvature (shallow arch) with load applied perpendicular to column**

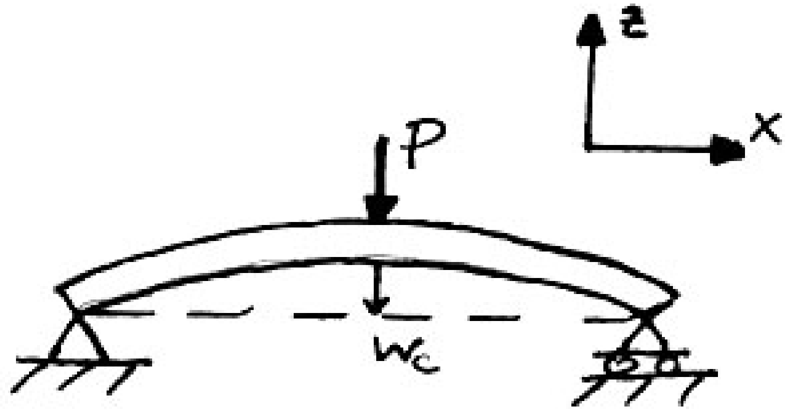
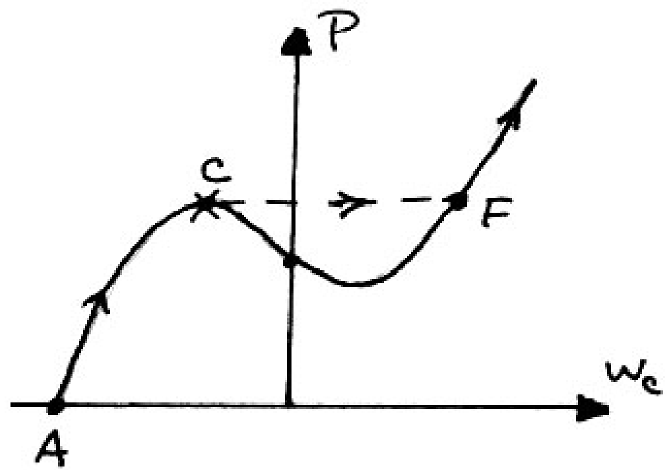


Figure 16.4 **Basic load-deflection behavior of shallow arch with transverse load**

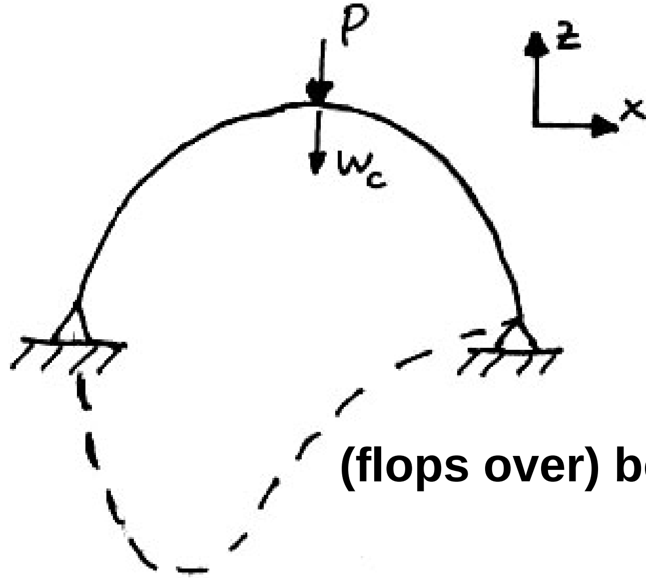


arch “snaps through” to F when load reaches C

Thus, there are nonlinear load-deflection curves in this behavior

For “deeper” arches, antisymetric behavior is possible

Figure 16.5 **Representation of antisymetric buckling of deeper arch under transverse load**

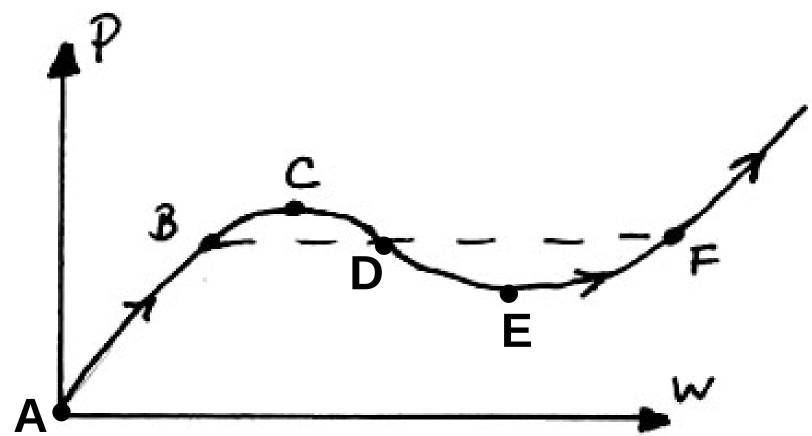


(flops over) before snapping through

Figure 16.6 **Load-deflection behavior of deeper arch under transverse load**

ABCDEF - symmetric snap-through

ABF - antisymmetric behavior



Will deal mainly with…

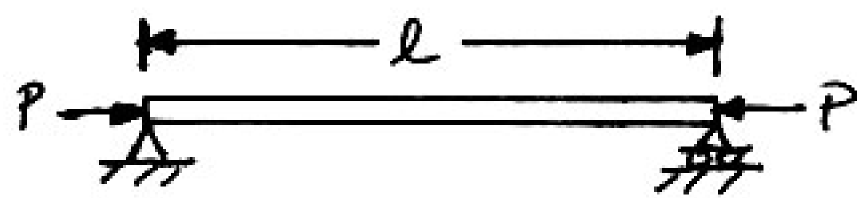
Bifurcation Buckling

First consider the "perfect" case: uniform column under end load.

First look at the simply-supported case…column is initially straight

- Load is applied along axis of beam
- "Perfect" column ⇒ only axial shortening occurs (before instability), i.e., no bending

Figure 16.7 **Simply-supported column under end compressive load**



EI = constant

Recall the <u>governing</u> <u>equation</u>:

$$EI \frac{d^4 w}{dx^4} + P \frac{d^2 w}{dx^2} = 0$$

--> Notice that P does not enter into the equation on the right hand side (making the differential equation homogenous), but enters as a coefficient of a linear differential term

This is an <u>eigenvalue</u> problem. Let:

$$w = e^{\lambda x}$$

this gives:

$$\lambda^4 + \frac{P}{EI} \lambda^2 = 0$$

$$\Rightarrow \lambda = \pm \sqrt{\frac{P}{EI}}\, i \qquad \underbrace{0,\ 0}$$

repeated roots $\Rightarrow$ need to look for more solutions

End up with the following <u>general</u> <u>homogenous</u> <u>solution</u>:

$$w = A \sin \sqrt{\frac{P}{EI}}\, x + B \cos \sqrt{\frac{P}{EI}}\, x + C + Dx$$

where the constants A, B, C, D are determined by using the Boundary Conditions

For the simply-supported case, boundary conditions are:

@ x = 0 $\begin{cases} w = 0 \\ M = EI\dfrac{d^2w}{dx^2} = 0 \end{cases}$

@ x = ℓ $\begin{cases} w = 0 \\ M = 0 \end{cases}$

From:

$$\left.\begin{array}{l} w(x = 0) = 0 \Rightarrow B + C = 0 \\ M(x = 0) = 0 \Rightarrow -EI\dfrac{P}{EI}B = 0 \end{array}\right\} \Rightarrow \begin{array}{l} B = 0 \\ C = 0 \end{array}$$

$$\left.\begin{array}{l} w(x = \ell) = 0 \Rightarrow A\sin\sqrt{\dfrac{P}{EI}}\,l + Dl = 0 \\ M(x = \ell) = 0 \Rightarrow -EI\dfrac{P}{EI}A\sin\sqrt{\dfrac{P}{EI}}\,l = 0 \end{array}\right\} \Rightarrow D = 0$$

and can see that:

$$A P \sin \sqrt{\frac{P}{EI}}\, l \;=\; 0$$

This occurs if:

- P = 0 (not interesting)
- A = 0 (trivial solution, w = 0)
- $\sin \sqrt{\frac{P}{EI}}\, l \;=\; 0$

$$\Rightarrow \quad \sqrt{\frac{P}{EI}}\, l \;=\; n\pi$$

Thus, the critical load is:

$$\boxed{P \;=\; \frac{n^2 \pi^2 EI}{l^2}}$$

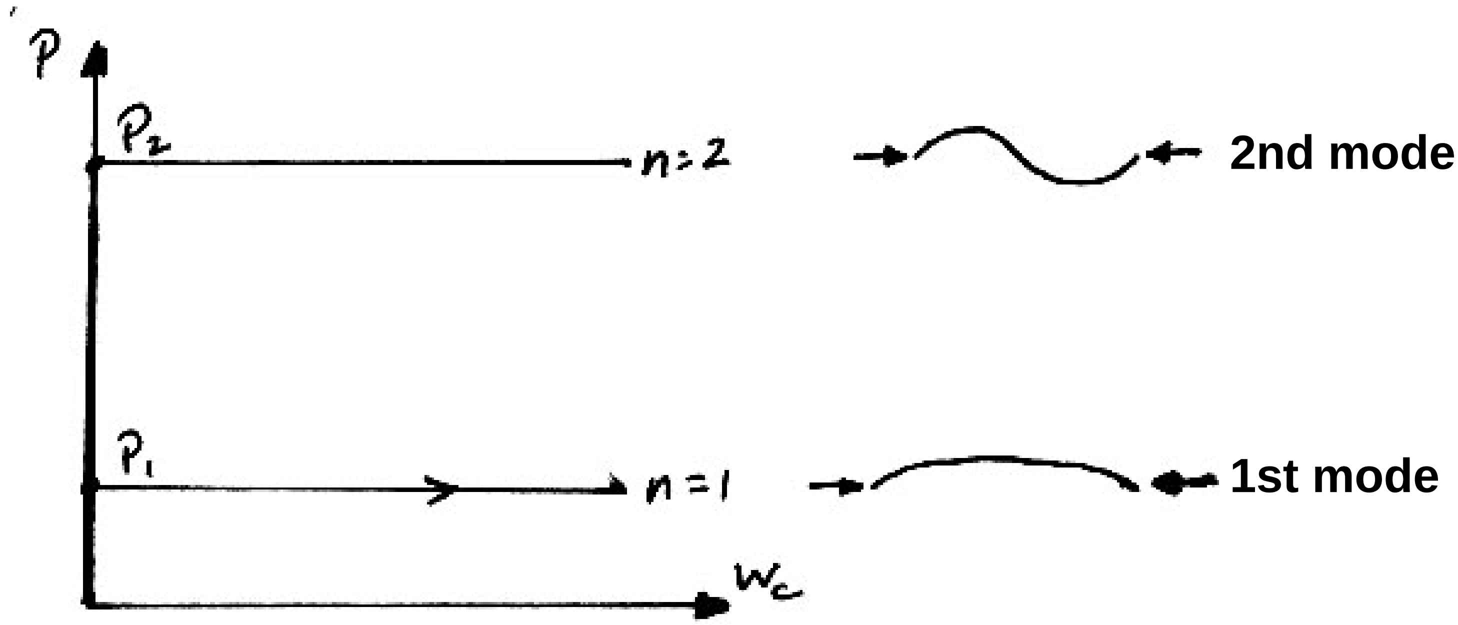
associated with each is a shape (*mode*)

$$w \;=\; A \sin \frac{n\pi x}{l}$$

A is still undefined (instability ⇒ w --> ∞)

So have buckling loads and associated mode shapes

Figure 16.8 **Representation of buckling loads and modes for simply-supported columns**



$P_2 = 4P_1$

The lowest value is the one where buckling occurs:

$$\boxed{P_{cr} = \frac{\pi^2 EI}{l^2}}$$

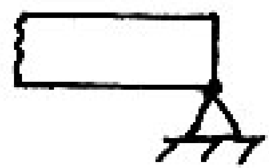
<u>Euler</u> <u>buckling</u> <u>load</u>

The higher loads can be reached if the column is "artificially restrained" at lower bifurcation points.

Other Boundary Conditions

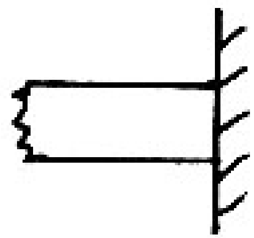
There are 3 (/4) allowable boundary conditions for w (need two on each end) which are homogeneous (⇒ … = 0)

- Simply-supported (pinned)



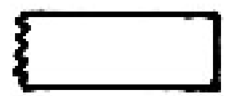
$$\begin{cases} w = 0 \\ M = EI \dfrac{d^2 w}{dx^2} = 0 \end{cases}$$

- Fixed end (clamped)



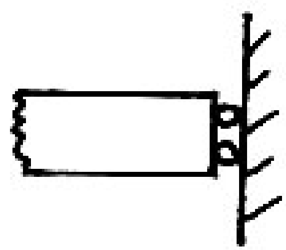
$$\begin{cases} w = 0 \\ \dfrac{dw}{dx} = 0 \end{cases}$$

- Free end



$$\begin{cases} M = EI \dfrac{d^2 w}{dx^2} = 0 \\ S = \dfrac{d}{dx}\left(EI \dfrac{d^2 w}{dx^2}\right) = 0 \end{cases}$$

- Sliding

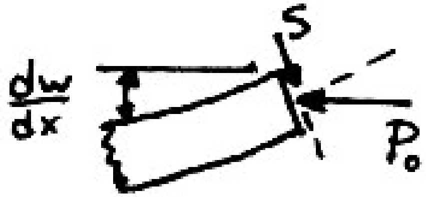


$$\begin{cases} S = 0 \\ \dfrac{dw}{dx} = 0 \end{cases}$$

There are others of these that are homogeneous and inhomogeneous Boundary Conditions

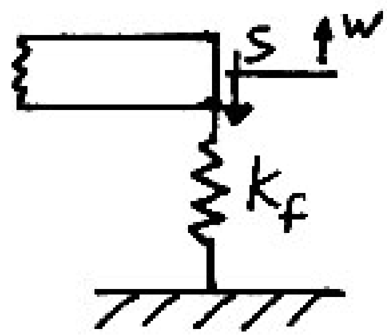
Examples:

- Free end with an axial load



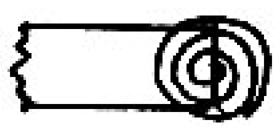
$$\begin{cases} M = 0 \\ S = -P_0 \dfrac{dw}{dx} \end{cases}$$

- Vertical spring



$$\begin{cases} M = 0 \\ S = k_f\, w \end{cases}$$

- Torsional spring



$$\begin{cases} w = 0 \\ M = -k_T \dfrac{dw}{dx} \end{cases}$$

Solution Procedure for P_{cr} :

- Use boundary conditions to get four equations in four unknowns (the constants A, B, C, D)
- Solve this set of equations to find non-trivial value of P

$$\underbrace{\begin{bmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{bmatrix}}_{\textbf{matrix}} \begin{Bmatrix} A \\ B \\ C \\ D \end{Bmatrix} = \mathbf{0}$$ **homogeneous equation**

- Set determinant of matrix to zero ($\Delta = 0$) and solve resulting equation.

Will find, for example, that for a clamped-clamped column:

$$P_{cr} = \frac{4\pi^2 EI}{l^2}$$ (need to do solution geometrically)

with the associated eigenfunction $\left(1 - \cos\frac{2\pi x}{l}\right)$

Figure 16.9 **Representation of clamped-clamped column under end load**

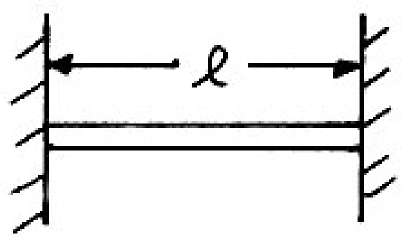


Figure 16.10 **Representation of buckling mode of clamped-clamped column**

Note terminology:

buckling load = eigenvalue

buckling mode = eigenfunction

Notice that this critical load has the same form as that found for the simply-supported column except it is multiplied by a factor of **4**

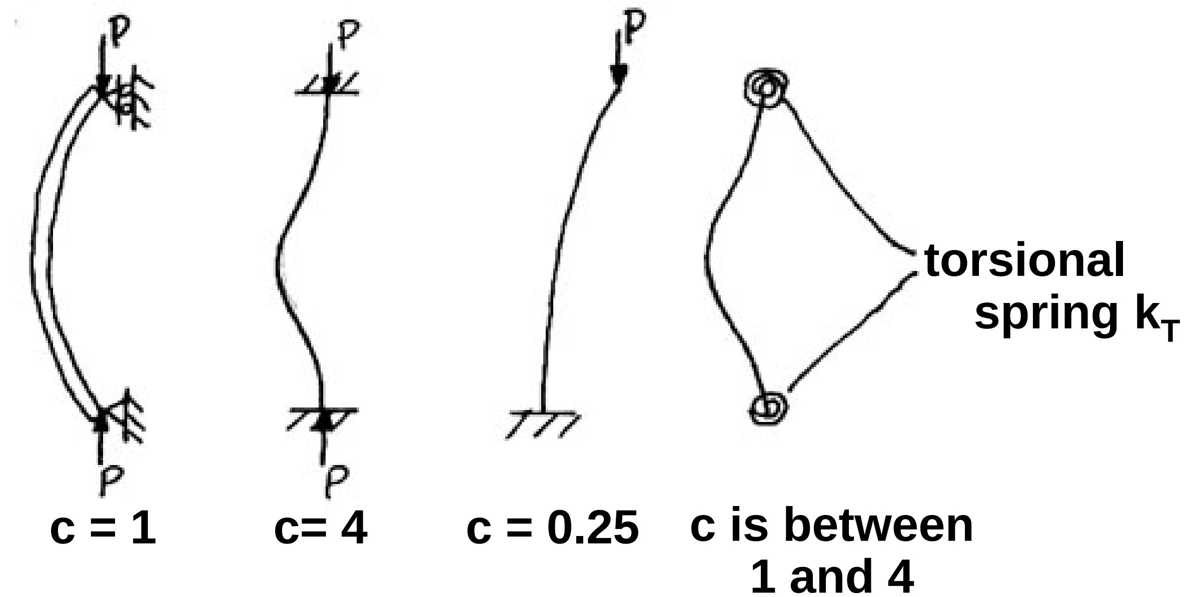
Can express the critical buckling load in the generic case as:

$$\boxed{P_{cr} = \frac{c\pi^2 EI}{l^2}}$$

where:

c = coefficient of edge fixity

Figure 16.11 **Representation of buckling of columns with different end conditions**



Generally use $c \approx 2$ for aircraft work with "fixed ends"

- Cannot truly get perfectly clamped end
- Simply-supported is too conservative
- Empirically, c = 2 works well and remains conservative

Other important parameters:

radius of gyration = $\rho = (I/A)^{1/2}$

slenderness ration = L/ρ

effective length = $L' = \dfrac{L}{\sqrt{c}}$

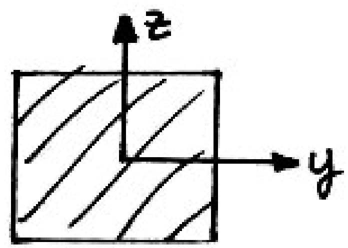
See Rivello

Considerations for Orthotropic or Composite Beams

If maintain geometrical restrictions of a column (ℓ >> in-plane directions), only the longitudinal properties, EI, are important. Thus, use techniques developed earlier:

- E_L for orthotropic
- $E_1 I^*$ for composite

Note: Consider general cross-section



Buckling could occur in y or z direction (or any direction transverse to x, for that matter).

--> must evaluate I^* for each direction and see which is less…buckling occurs for the case where I^* is smaller

--> anywhere in y-z plane

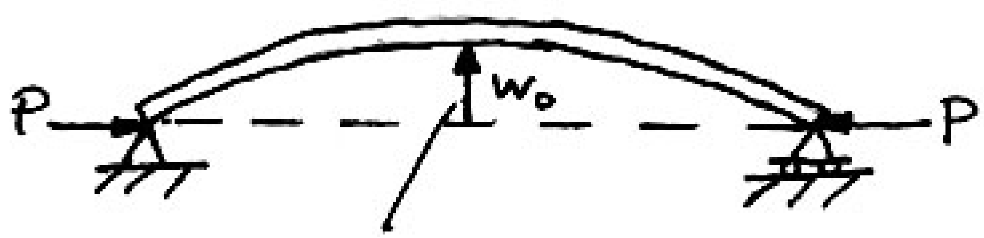
--> use Mohr's circle

Note: May need to be corrected for shearing effects

See Timoshenko and Gere, Theory of Elastic Stability, pp. 132-135

Effects of Initial Imperfections

Figure 16.12 **Representation of column with initial imperfection**



initial deflection (imperfection due to manufacturing, etc.)

OR

Figure 16.13 **Representation of column loaded eccentrically**

e = "eccentricity"

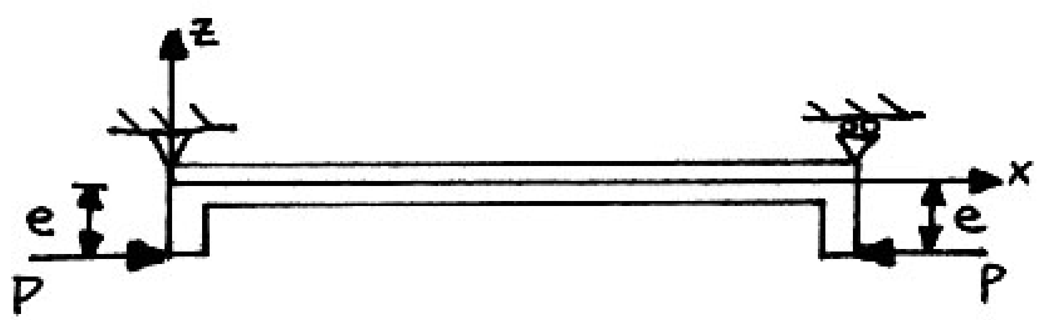
load <u>not</u> applied along center line of column

These two cases are basically handled the same -- a moment is applied in each case

- Case 1 -- due to initial imperfection
- Case 2 -- since load is not applied along axis of column (beam)

Look closely at second case:

Figure 16.14 **Representation of full geometry of simply-supported column loaded eccentrically**



The governing equation is still the same:

$$EI \frac{d^4 w}{dx^4} + P \frac{d^2 w}{dx^2} = 0$$

and thus the basic solution is the same:

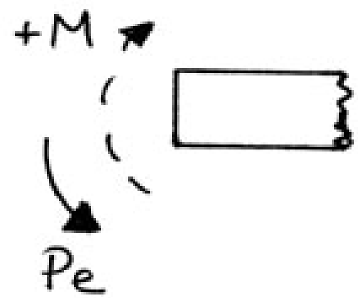
$$w = A \sin \sqrt{\frac{P}{EI}}\, x + B \cos \sqrt{\frac{P}{EI}}\, x + C + Dx$$

What changes are the <u>Boundary</u> <u>Conditions</u>

For the specific case of Figure 16.14:

$$@\ x = 0 \quad \left. \begin{array}{l} \left\{ \begin{array}{l} w = 0 \quad \text{-->} \quad B + C = 0 \\ M = EI \dfrac{d^2 w}{dx^2} = -Pe \end{array} \right. \\ \Rightarrow -EI \dfrac{P}{EI} B = -Pe \end{array} \right\} \Rightarrow \begin{array}{l} B = e \\ C = -e \end{array}$$

Figure 16.15 **Representation of end moment for column loaded eccentrically**



$$@\ x = \ell \quad \begin{cases} w = 0 \Rightarrow A \sin\sqrt{\frac{P}{EI}}\, l + e \cos\sqrt{\frac{P}{EI}}\, l - e + Dl = 0 \\ M = EI \frac{d^2 w}{dx^2} = -Pe \Rightarrow \end{cases}$$

$$-EI \frac{P}{EI} A \sin\sqrt{\frac{P}{EI}}\, l - EI \frac{P}{EI} e \cos\sqrt{\frac{P}{EI}}\, l = -Pe$$

Find: D = 0

$$A = \frac{e\left(1 - \cos\sqrt{\frac{P}{EI}}\, l\right)}{\sin\sqrt{\frac{P}{EI}}\, l}$$

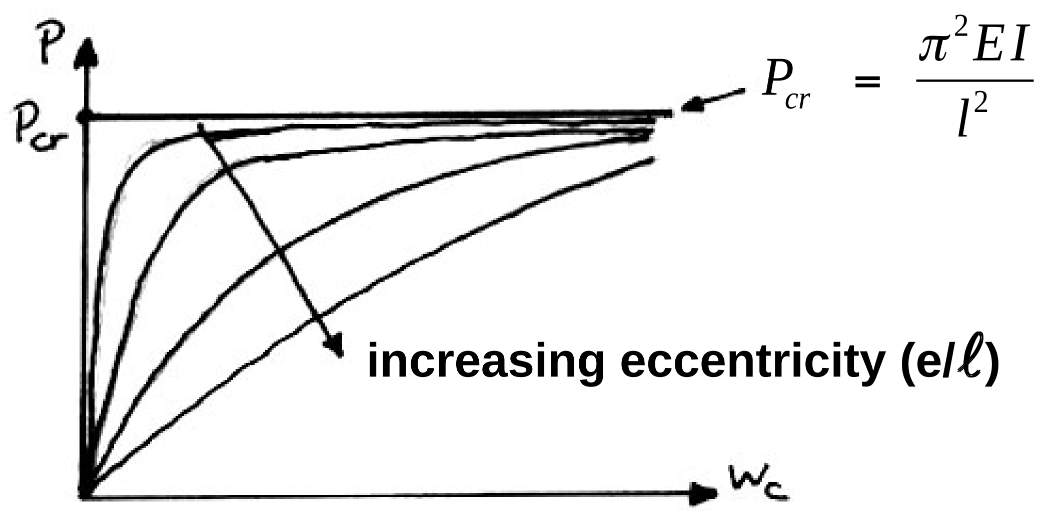
Putting this all together, find:

$$\boxed{w = e\left\{\frac{\left(1 - \cos\sqrt{\frac{P}{EI}}\, l\right)}{\sin\sqrt{\frac{P}{EI}}\, l} \sin\sqrt{\frac{P}{EI}}\, x + \cos\sqrt{\frac{P}{EI}}\, x - 1\right\}}$$

Deflection is generally finite (this is not an eigenvalue problem).

However, as P approaches $P_{cr} = \frac{\pi^2 EI}{l^2}$, w again becomes unbounded (w --> ∞)

Figure 16.16 **Load-deflection response for various levels of eccentricity of end-loaded column**



--> Nondimensional problem via e/ℓ

So, w approaches perfect case as P approaches P_{cr}. But, as e/ℓ increases, behavior is less like perfect case.

Bending Moment now:

$$M = EI\frac{d^2w}{dx^2} = -eP\left\{\frac{\left(1 - \cos\sqrt{\frac{P}{EI}}\,l\right)}{\sin\sqrt{\frac{P}{EI}}\,l}\sin\sqrt{\frac{P}{EI}}\,x + \cos\sqrt{\frac{P}{EI}}\,x\right\}$$

As P goes to zero, M --> -eP

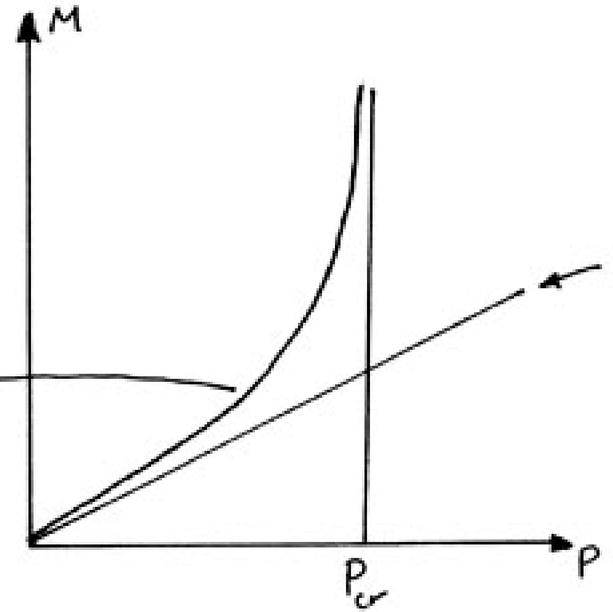
This is known as the primary bending moment (i.e., the bending moment due to axial loading)

Also note that as $\sqrt{\frac{P}{EI}}\,l$ --> π (P --> P_{cr}), M --> ∞

(This is due to the fact that there is an instability as w --> ∞. This cannot happen in real life)

Figure 16.17 **Moment-load response for eccentrically loaded column**

actual behavior is bonded by two asymptotes



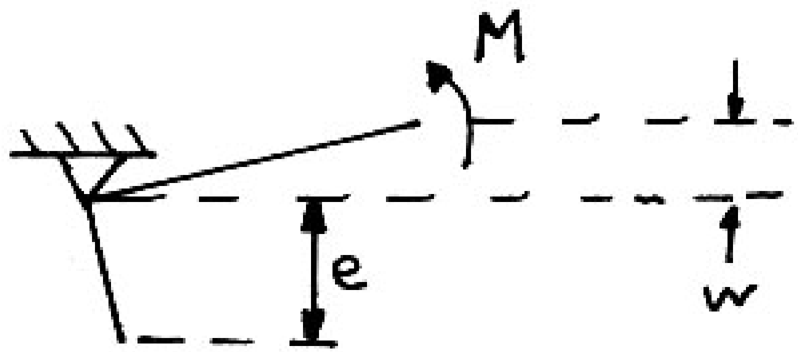
Overall:

Primary
Bending Moment

$$M = -Pe - Pw$$

Secondary (due to deflection)
Bending Moment

Figure 16.18 **Representation of moments due to eccentricity and deflection**



<u>Note</u>: All this is good for small deflections. As deflections get large, have <u>post</u> <u>buckling</u> considerations.
(Will discuss later)