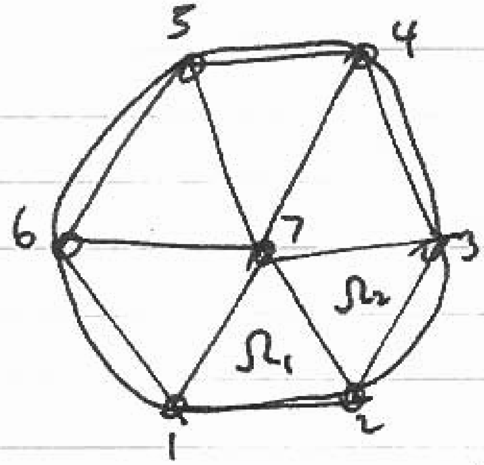


① $B \sim B_h = \bigcup_{e=1}^{E} \Omega_h^e$

② Use local interpolation of u_h

$u_h^e \equiv$ restriction of u_h to Ω_h^e

$$u_h^e = \sum_{a=1}^{n} u_a^e N_a^e(x)$$

- $N_a^e(x_b^e) = \delta_{ab}$

Global interpolation: Local element nodes must "fit" together and define global nodes.

- Continuity requirements

Global "u_h" must be in $H_0^1(B)$

Math aside:

Need to measure "size" of functions (errors in particular) $\Longrightarrow$ norms and seminorms.

Natural norms to use in problems such as linear elasticity:

Sobolev norms

Need convenient way to express partial derivatives: <u>multi-indices</u>

Multi-index "α" of dimension "d" is an array of nonnegative indices:

$$\{\alpha_1, \alpha_2, \ldots, \alpha_d\}$$

The degree $|\alpha|$ of the multi-index is the sum

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$$

<u>Definition:</u> $u : \mathbb{R}^d \to \mathbb{R}$

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_d^{\alpha_d}}$$

<u>Definition:</u> <u>Seminorm:</u>

Let $\Omega \in \mathbb{R}^d$ an open bounded set, $m \geq 0$, $1 \leq p < \infty$

$u : \Omega \to \mathbb{R}$ m-times continuously differentiable in Ω. $(C^m(\Omega))$

$$\boxed{|u|_{m,p} = \left(\sum_{|\alpha| = m} \int_\Omega |D^\alpha u|^p \, dx \right)^{1/p}}$$

<u>Definition</u> $\Omega \in \mathbb{R}^d$ bounded open set, $m \geq 0$, $1 \leq p < \infty$

$u : \Omega \to \mathbb{R}$ $C^m(\Omega)$. Norm:

$$\boxed{\|u\|_{m,p} = \left(\sum_{k=0}^{m} |u|_{k,p}^{p} \right)^{1/p}}$$

<u>Definition:</u> $W^{m,p}(\Omega)$ the Sobolev space of

functions which can be obtained as limits of smooth functions under the norm $\|\cdot\|_{m,p}$.

Roughly speaking, these limits may be thought as

functions in $L^p(\Omega)$ whose derivatives (in the distributional sense) up to order "m" are themselves in $L^p(\Omega)$. In particular, the space

$$W^{0,p} = L^p(\Omega) \quad \text{Lebesgue space.}$$

Following standard practice, we shall denote

$$H^m(\Omega) \equiv W^{m,2}(\Omega)$$

The Sobolev space $W^{m,p}$ is a complete normed space. (Banach space).

In addition, $H^m(\Omega)$ are Hilbert spaces with the inner product:

$$(u, v)_m = \sum_{|\alpha| \leq m} \int_\Omega D^\alpha u \cdot D^\alpha v \, dx$$

end aside

Want $\underline{u_h \in H_0^1(B)}$ $\rightarrow u = \bar{u}$ on S_1

$H_0^1(B) = \{ u : B \subset \mathbb{R}^d \rightarrow \mathbb{R}^d \, / \, \|u\|_{1,2} < \infty, u = u_0 \text{ on } S_1 \}$

Think in terms of:

$$J(u) = \int_B \frac{1}{2} C_{ijkl} u_{k,l} u_{i,j} \, dV - \int_B f_i u_i \, dv - \int_{S_2} t_i u_i \, ds$$

$$\|u\|_E = \sqrt{a(u,u)} = \sqrt{\int_B C_{ijkl} u_{k,l} u_{i,j} \, dV}$$

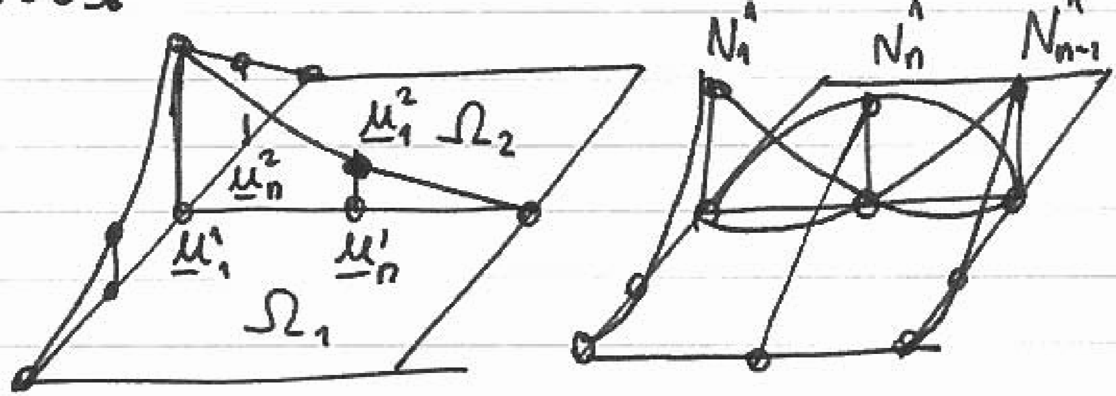
<u>Conditions on u_h^e (local interpolation)</u>

① N_a^e must be $C^1(\Omega_h^e)$ (sufficient, not necessary)

② Global shape functions obtained by piecing together local shape functions must be C^0:

derivatives may jump on a set of measure "0"

Shape functions N_a^e must be uniquely defined on sides:



<u>Global shape function:</u>

$$u_h(x) = \sum_{e=1}^{E} u_h^e(x) = \sum_{e=1}^{E} \sum_{a=1}^{n} N_a^e(x)\, u_{ia}^e$$

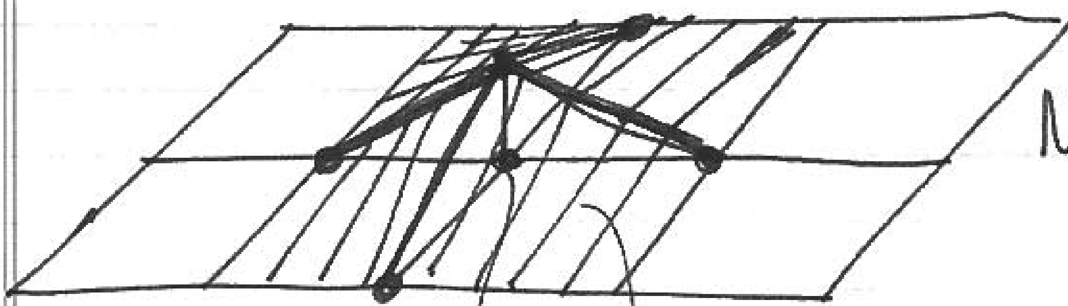
(x not in boundary of elements)

through connectivity map: $g(b,e) = a$

$a = 1, \dots, N$
$b = 1, \dots, n$
$e = 1, \dots, E$

$$\left.\begin{aligned} X_{g(b,e)} &= X_a^e \\ u_{g(b,e)} &= u_a^e \end{aligned}\right| \rightarrow \text{global/local mapping}$$

$$u_h(x) = \sum_{e=1}^{E} \sum_{b=1}^{n} N_b^e(x)\, \underbrace{u_{g(b,e)}}_{u_a} = \sum_{a=1}^{N} u_a N_a(x)$$



$N_a = \sum_e N_a^e$

a

N_a : compact support

$$\text{support}(N_a) = \{ \Omega_s \subset B \, / \, N_a(x) \neq 0 \;\; \forall x \in \Omega_s \}$$

$$= \{ \cup \, \Omega^e \text{ incident to node "a"} \}$$

Computation of K and f^{ext}

$$K_{iakb} = \int_B C_{ijkl} \, N_{a,j} \, N_{b,l} \, dV$$

$$= \int_B C_{ijkl} \left(\sum_{e=1}^{E} N^e_{a,j} \right) \left(\sum_{f=1}^{E} N^f_{b,l} \right) dV$$

$$= \sum_{e,f=1}^{E} \int_B C_{ijkl} \, N^e_{a,j} \, N^f_{b,l} \, dV$$

N_a^e compact support $\quad \sum_f \sum_e \to \sum_e \, , \; \int_B = \int_{\Omega^e}$

$$= \sum_{e=1}^{E} \underbrace{\boxed{\int_{\Omega^e} C_{ijkl} \, N^e_{a,j} \, N^e_{b,l} \, dV}}_{K^e}$$

$$\boxed{K_{iakb} = \sum_{e=1}^{E} K^e}$$

↑ assembly operator

Similarly: $f_{ia}^{ext} = \int_B f_i N_a \, dv + \cancel{\text{tractions}}$ neglect for the moment.

$$= \int_B f_i \left(\sum_e N_a^e\right) dv = \sum_e \underbrace{\int_{\Omega^e} f_i N_a^e \, dv}_{\left(f_{ia}^{ext}\right)^e}$$

$$\boxed{f_{ia}^{ext} = \sum_e \left(f_{ia}^{ext}\right)^e}$$

↑ assembly operation.

Skip "B" matrix.

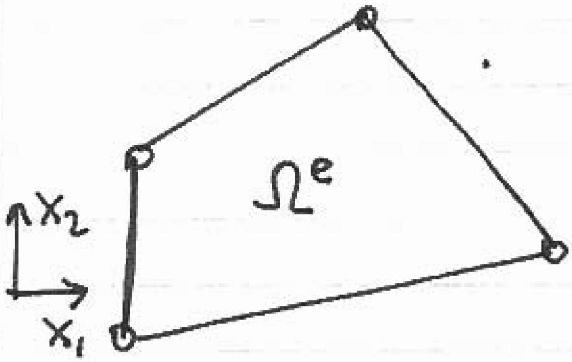
Isoparametric elements

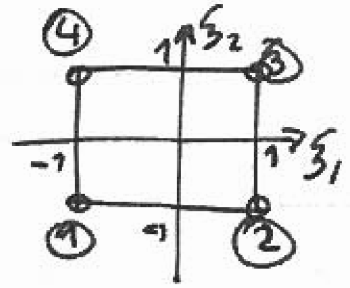
Lagrangian family (quadrilaterals, hexahedra)

Ref. "Finite element procedures in engineering analysis" K.J. Bathe, Prentice Hall, 2nd edition (1995)

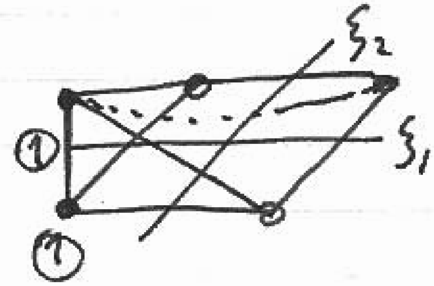
"The finite element Method" T.J.R. Hughes, Dover 2000 ← Linear static and dynamic analysis

"The finite element method" O.C. Zienkiewicz, R.L. Taylor 5th edition, 2000





Define standard shape functions on standard domain (low order polynomials)



$$\hat{N}_1(\xi_1,\xi_2) = \frac{1}{4}(1-\xi_1)(1-\xi_2)$$

$$\hat{N}_2(\xi_1,\xi_2) = \frac{1}{4}(1+\xi_1)(1-\xi_2)$$

$$\hat{N}_3(\xi_1,\xi_2) = \frac{1}{4}(1+\xi_1)(1+\xi_2)$$

$$\hat{N}_4(\xi_1,\xi_2) = \frac{1}{4}(1-\xi_1)(1+\xi_2)$$

L5

Verify: i) $\hat{N}_a(\xi_b) = \delta_{ab}$
ii) Conformity (C^0): restrictions of $\hat{N}_a$ to element sides are linear

Define $N_a(x_i)$: Isoparametric mapping