Material formulation

$$f_{ia}^{int} = \sum_e \int_{\Omega_0^e} P_{iJ} \, N_{a,J} \, dV_0$$

$$K_{iakb} = \sum_e \int_{\Omega_0^e} C_{iIjJ} \, N_{b,J} \, N_{a,I} \, dV_0$$

Specific material models

Isotropic elasticity:

$$W = W(C) \longrightarrow \text{isotropy} \quad W = W(I_1, I_2, I_3)$$

where I_1, I_2, I_3 are the invariants of C resulting from the characteristic equation:

$$\det(C - \lambda I) = \lambda^3 - I_1 \lambda^2 + I_2 \lambda - I_3$$

$$\begin{cases} I_1 = \operatorname{tr} C \\ I_2 = \frac{1}{2}\left[\operatorname{tr}^2 C - \operatorname{tr} C^2\right] \\ I_3 = \det C \end{cases}$$

Stress-strain relations

$$S_{IJ} = 2\frac{\partial W}{\partial C_{IJ}} = 2\left[\frac{\partial W}{\partial I_1}\frac{\partial I_1}{\partial C_{IJ}} + \frac{\partial W}{\partial I_2}\frac{\partial I_2}{\partial C_{IJ}} + \frac{\partial W}{\partial I_3}\frac{\partial I_3}{\partial C_{IJ}}\right]$$

- $$\frac{\partial I_1}{\partial C_{IJ}} = \frac{\partial}{\partial C_{IJ}}(C_{11}+C_{22}+C_{33}) = \delta_{I1}\delta_{J1}+\delta_{I2}\delta_{J2}+\delta_{I3}\delta_{J3}$$
$$= \delta_{IJ}$$

- $$\frac{\partial I_2}{\partial C_{IJ}} = \frac{\partial}{\partial C_{IJ}}\left[\frac{1}{2}\left(I_1^2 - tr\, C^2\right)\right] = I_1\frac{\partial I_1}{\partial C_{IJ}} - \frac{1}{2}\frac{\partial\, tr C^2}{\partial C_{IJ}}$$

$$= I_1\delta_{IJ} - \frac{1}{2}\frac{\partial}{\partial C_{IJ}}\left(C_{11}^2 + C_{12}C_{21} + C_{13}C_{31} + C_{21}C_{12} + C_{22}^2 + \right.$$

$$\left. + C_{23}C_{32} + C_{31}C_{13} + C_{32}C_{23} + C_{33}^2\right)$$

$$= I_1\delta_{IJ} - \left(C_{11}\delta_{I1}\delta_{J1} + C_{12}\delta_{I2}\delta_{J1} + C_{13}\delta_{I3}\delta_{J1} + \right.$$

$$C_{21}\delta_{I1}\delta_{J2} + C_{22}\delta_{I2}\delta_{J2} + C_{23}\delta_{I3}\delta_{J2} +$$

$$\left. C_{31}\delta_{I1}\delta_{J3} + C_{32}\delta_{I2}\delta_{J3} + C_{33}\delta_{I3}\delta_{J3}\right)$$

$$= I_1\delta_{IJ} - C_{IJ}$$

$$\frac{\partial I_3}{\partial C_{IJ}} = I_3\, C^{-1}_{IJ}$$

$$S_{IJ} = 2\Bigg[\underbrace{\left(\frac{\partial W}{\partial I_1} + I_1 \frac{\partial W}{\partial I_2}\right)}_{A_0} \delta_{IJ} - \underbrace{\frac{\partial W}{\partial I_2}}_{A_1} C_{IJ} + \underbrace{\frac{\partial W}{\partial I_3} I_3}_{A_2} C^{-1}_{IJ}\Bigg]$$

$$S_{IJ} = A_0\, \delta_{IJ} + A_1\, C_{IJ} + A_2\, C^{-1}_{IJ}$$

Cayley-Hamilton theorem: $C^3 - I_1 C^2 + I_2 C - I_3 I = 0$

$$C^{-1} = \frac{1}{I_3}\left(C^2 - I_1 C + I_2 I\right)$$

Alternative form:

$$\boxed{S_{IJ} = B_0\, \delta_{IJ} + B_1\, C_{IJ} + B_2\, C_{IK}\, C_{KJ}}$$

- Exercise: Express B_i, $i = 0, 2$ in terms of W

Spatial form: $\tau = F S F^T$, $\sigma = J^{-1} F S F^T$

$$S_{IJ} = B_0 \delta_{IJ} + B_1 F_{kI} F_{kJ} + B_2 F^{-1}_{Jk} F^{-1}_{Ik}$$

$$\tau_{ij} = F_{iI} S_{IJ} F_{jJ} =$$

$$B_0 \underbrace{F_{iI} F_{jI}}_{b_{ij}} + B_1 \underbrace{F_{iI} F_{kI}}_{b_{ik}} \underbrace{F_{kJ} F_{jJ}}_{b_{kj}} + B_2 \underbrace{\delta_{jk} \delta_{ik}}_{\delta_{ij}}$$

$b = F F^T \equiv$ left Cauchy-Green deformation tensor

$$\Rightarrow \boxed{\sigma_{ij} = \alpha_0 \delta_{ij} + \alpha_1 b_{ij} + \alpha_2 b^{-1}_{ij}}$$

$$\boxed{\sigma_{ij} = \beta_0 \delta_{ij} + \beta_1 b_{ij} + \beta_2 b_{ik} b_{kj}}$$

Examples of constitutive relations for finite elasticity

1) Saint-Venant/Kirchhoff model

$$S_{IJ} = \lambda E_{KK} \delta_{IJ} + 2\mu E_{IJ}$$

λ, μ constants

It works well for moderate deformations

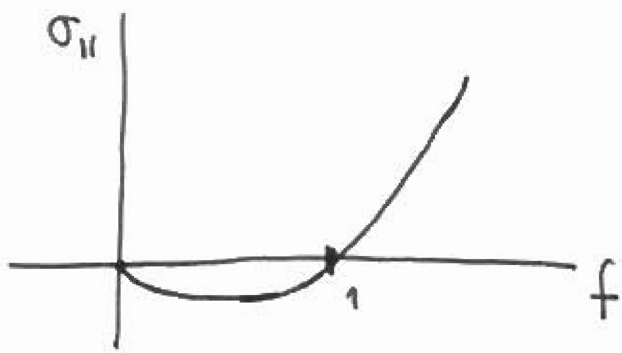
but it has the wrong limit.

1D deformation
$$F = \begin{pmatrix} \overbrace{l/L_0}^{f} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} (l/L_0)^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad E = \begin{pmatrix} \frac{1}{2}(l/L_0)^2 - 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$S_{11} = (\lambda + 2\mu)\, E_{11}$$

$$\sigma_{11} = f\, S_{11} = (\lambda + 2\mu)\, \frac{f}{2}\,(f^2 - 1)$$



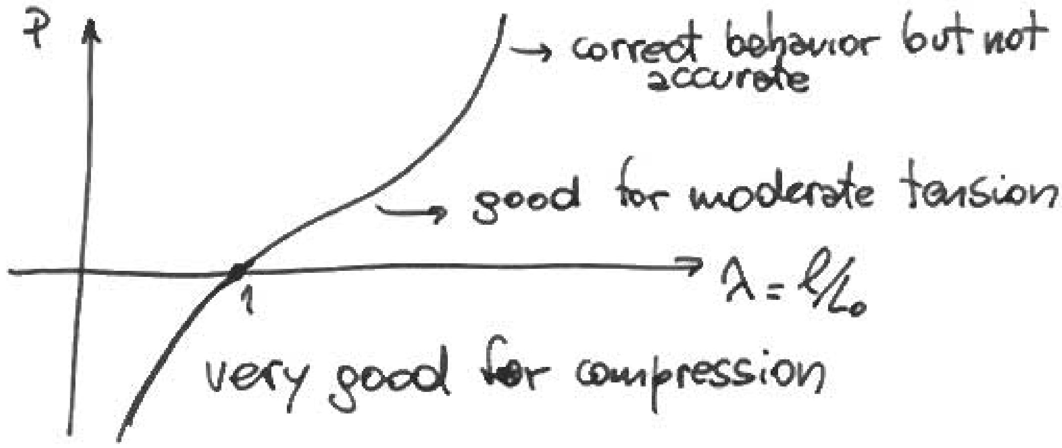
for $l = 0$, $\sigma_{11} = 0$

VERY BAD!!

2) Mooney-Rivlin (incompressible)

$$\sigma_{ij} = p\,\delta_{ij} + \alpha_1\, b_{ij} - \alpha_2\, b^{-1}_{ij} \; ; \quad \alpha_1 \geq 0 \,, \; \alpha_2 \geq 0$$

Potential:
$$W(C) = \frac{1}{2}\left[\alpha_1\,(I_1 - 3) + \alpha_2\,(I_2 - 3)\right]$$



3) Neo-Hookean model extended to compressible range

$$W(C) = \underbrace{\frac{\lambda_0}{2} \log^2 J - \mu_0 \log J}_{\text{compressibility}} + \underbrace{\frac{\mu_0}{2} I_1}_{\text{Neo-Hookean}}$$

$$S_{IJ} = \frac{2\,\partial W}{\partial C_{IJ}} = 2\lambda_0 \log J \,\frac{1}{J}\,\frac{\partial J}{\partial C_{IJ}} - \frac{2\mu_0}{J}\frac{\partial J}{\partial C_{IJ}} + \mu_0 \delta_{IJ}$$

$$J = \det(F) = \sqrt{\det(C)}$$

$$\frac{\partial J}{\partial C_{IJ}} = \frac{1}{2}\frac{1}{\sqrt{\det(C)}}\,\frac{\partial \det(C)}{\partial C_{IJ}} = \frac{1}{2}\frac{1}{\sqrt{\det(C)}} \det(C)\, C^{-1}_{IJ}$$

$$= \frac{J}{2}\, C^{-1}_{IJ}$$

$$S_{IJ} = \cancel{2}\lambda_0 \log J \frac{1}{\cancel{J}} \frac{\cancel{J}}{\cancel{2}} C^{-1}_{IJ} - \frac{\cancel{2}\mu_0}{\cancel{J}} \frac{\cancel{J}}{\cancel{2}} C^{-1}_{IJ} + \mu_0 \delta_{IJ}$$

$$= (\lambda_0 \log J - \mu_0)\, C^{-1}_{IJ} + \mu_0 \delta_{IJ}$$

$$\boxed{S_{IJ} = \lambda_0 \log J \; C^{-1}_{IJ} + \mu_0 \left(\delta_{IJ} - C^{-1}_{IJ}\right)}$$

push forward to spatial configuration:

$$\sigma_{ij} = J^{-1} S_{IJ} F_{iI} F_{jJ}$$

$$\boxed{J\,\sigma_{ij} = \lambda_0 \log J \; \delta_{ij} + \mu_0 \left(b_{ij} - \delta_{ij}\right)}$$

Infinitesimal:

$$b_{ij} = F_{iI} F_{jI} = (\delta_{iI} + u_{i,I})(\delta_{jI} + u_{j,I})$$

$$= \delta_{ij} + u_{j,i} + u_{i,j} + \underbrace{\cancel{u_{i,I} u_{j,I}}}_{\text{h.o.t.}}$$

$$\sim \delta_{ij} + 2\varepsilon_{ij} \quad , \; \varepsilon_{ij} \equiv \text{small strain tensor}$$

Similarly $J \sim 1 + \varepsilon_{kk}$

$$\boxed{\sigma_{ij} \sim \lambda_0 \varepsilon_{kk} \delta_{ij} + 2\mu_0 \varepsilon_{ij}} \quad \text{Hooke's law}$$

can be used to measure λ_0, μ_0: initial Lamé constants.

Computation of tangent moduli

$$C_{IJKL} = 2\frac{\partial S_{IJ}}{\partial C_{KL}} = 4\frac{\partial^2 W}{\partial C_{IJ}\partial C_{KL}}$$

$$= 2\left\{ \lambda_0 \frac{1}{J} \underbrace{\frac{\partial J}{\partial C_{KL}}}_{\frac{J}{2} C^{-1}_{KL}} C^{-1}_{IJ} + \lambda_0 \log J \underbrace{\frac{\partial C^{-1}_{IJ}}{\partial C_{KL}}}_{?} - \mu_0 \frac{\partial C^{-1}_{IJ}}{\partial C_{KL}} \right\}$$

$$C^{-1}_{IK} C_{KJ} = \delta_{IJ}$$

$$\frac{\partial C^{-1}_{IK}}{\partial C_{LM}} C_{KJ} + C^{-1}_{IK}\left(\delta_{KL}\delta_{JM} + \delta_{KM}\delta_{JL}\right)\frac{1}{2} = 0$$

$$\frac{\partial C^{-1}_{IK}}{\partial C_{LM}} C_{KJ} + \frac{1}{2}\left(C^{-1}_{IL}\delta_{JM} + C^{-1}_{IM}\delta_{JL}\right) = 0$$

$$\frac{\partial C^{-1}_{IK}}{\partial C_{LM}} \underbrace{C_{KJ} C^{-1}_{JN}}_{\delta_{KN}} = -\frac{1}{2}\left(C^{-1}_{IL} C^{-1}_{MN} + C^{-1}_{IM} C^{-1}_{NL}\right)$$

$$\boxed{\frac{\partial C^{-1}_{IN}}{\partial C_{LM}} = -\frac{1}{2}\left(C^{-1}_{IL} C^{-1}_{MN} + C^{-1}_{IM} C^{-1}_{NL}\right)}$$

$$C_{IJKL} = \lambda_0 C^{-1}_{IJ} C^{-1}_{KL} + 2(\lambda_0 \log J - \mu_0)\left(\frac{-1}{2}\right)\left(C^{-1}_{IK} C^{-1}_{JL} + C^{-1}_{IL} C^{-1}_{JK}\right)$$

$$\boxed{C_{IJKL} = \lambda_0 C^{-1}_{IJ} C^{-1}_{KL} + (\mu_0 - \lambda_0 \log J)\left(C^{-1}_{IK} C^{-1}_{JL} + C^{-1}_{IL} C^{-1}_{JK}\right)}$$

spatial: $c_{ijkl} = J^{-1} F_{iI} F_{jJ} F_{kK} F_{lL} C_{IJKL}$

$$\boxed{c_{ijkl} = \underbrace{\frac{\lambda_0}{J}}_{\lambda(J)} \delta_{ij} \delta_{kl} + \underbrace{\frac{(\mu_0 - \lambda_0 \log J)}{J}}_{\mu(J)} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})}$$

$$\boxed{\lambda(J) = \frac{\lambda_0}{J}} \quad , \quad \boxed{\mu(J) = \frac{\mu_0 - \lambda_0 \log J}{J}}$$

Simo et al: It is not possible to have constant material parameters and elastic behavior. If material parameters are constant there is inexorably dissipation.