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16.323 Principles of Optimal Control
Spring 2008

# 16.323 Homework Assignment #4

1. The derivation on pages 6–1 – 6–2 was done for the case of free or fixed $x(t_f)$ and then repeated on 6–3 for more general boundary condition $m(x(t_f), t_f) = 0$.

   (a) Use the result on 6–3 to derive an optimal controller for the same case as Example 6–1 (on page 6–5) but with terminal constraints that $\dot{y}(t_f) = 0$ and

   $$y^2(t_f) + (t_f - 5)^2 - 4 = 0$$

   (b) Use `fsolve` (as on 5–18) to solve for the unknown parameters (there should be 5) in the control problem, and then use a simulation of the system to confirm that the control inputs cause the system to reach the target set at the appropriate $t_f$.

      (i) Consider two cases: (a) $\alpha = 0.1$, $b = 1$; and (b) $\alpha = 100$, $b = 1$.

2. Show that if in the Hamiltonian $H$, $a$ and $g$ are independent of time $t$, then $H$ is a constant, and that if $t_f$ is free and $h$ is not an explicit function of time, then this constant is zero. This is an important result that can often be used to help solve the control problems.

3. Read the posted article by Betts "Survey of numerical methods for. trajectory optimization," AIAA J. of Guidance,. Control and Dynamics, 21:193-207, 1998, and write a 1 page summary of his suggestions/conclusions.

4. We discussed in class that LQR is a great **regulator** in that it quickly returns the system states to 0 while balancing the amount of control used. However, we are also interested in tracking a reference command, so that $y(t) = r(t)$ as $t \to \infty$.

   (a) Design a steady state LQR controller for the system using $R_{xx} = I_2$, $R_{uu} = 0.01$

   $$\dot{x} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

   A naive way to implement a reference tracker is to modify the LQR controller from $u = -Kx$ to $u = r - Kx$:

   $$\dot{x} = (A - BK)x + Br , \quad y = Cx$$

   Verify that this leads to particularly poor tracking of a step input!

(b) An alternative strategy is to use $u = Nr - Kx$, where $N$ is a constant. What is a good way to choose $N$ to ensure zero steady state error for this closed-loop system? What are the consequences of this change in the step response of the closed-loop system?

(c) A completely different approach to ensuring zero steady-state error is to use what is often called an LQ-servo. The approach is to add a new state to the system that integrates the tracking error: $\dot{x}_i = r - y = r - Cx$, giving:

$$\begin{bmatrix} \dot{x} \\ \dot{x}_i \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} \begin{bmatrix} x \\ x_i \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$$

$$y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ x_i \end{bmatrix}$$

The LQR problem statement can now be modified (ignore $r$ in the design of $u$) to place a high weighting on $x_i$ to penalize the tracking error. Use this technique to design a new controller (keep $R_{xx}$ and $R_{uu}$ the same as part(a) and tune the weight on $x_i$ to achieve a performance that is similar to part (b)). Compare the transient responses for the approaches in (b) and (c) - do you see any advantages to one approach over the other?

5. Consider the optimal control of thrust vector direction that maximizes the terminal velocity of the space launch vehicle in the rectilinear frame. The vehicle dynamics (see figure) is represented as

$$\begin{aligned} \dot{u} &= a\cos\beta \\ \dot{v} &= a\sin\beta \\ \dot{x} &= u \\ \dot{y} &= v \end{aligned}$$

where $(x, y)$ and $(u, v)$ denote the position and the velocity of the rocket with respect to an rectilinear inertial frame. The thrust acceleration $a$ is assumed to be a known function of time. The boundary conditions are given as:

$$\begin{aligned} u(t_0) &= v(t_0) = x(t_0) = y(t_0) = 0, \\ v(t_f) &= 0, \quad y(t_f) = h. \end{aligned}$$

Also, the terminal time $t_f$ is given and $h$ is the altitude of the target orbit.

(a) Show that the optimal control law takes the form of

$$\tan\beta = \tan\beta_0 - ct \tag{1}$$

where $\beta_0$ and $c$ are constant. This law is referred to as *linear tangent law*.

(b) What happens if there exists the gravity described as:

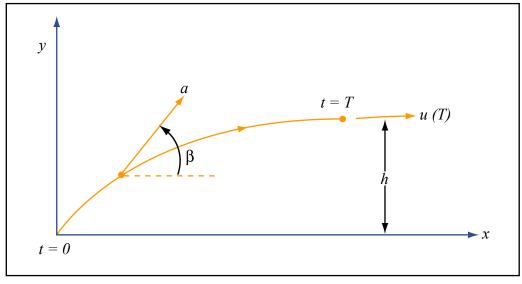$$\dot{v} = a\sin\theta - g \tag{2}$$

with $g$ being constant?

Figure by MIT OpenCourseWare.

Figure 1: Figure for Problem 4

(c) Use the numerical methods discussed in class to find the optimal control inputs and resulting path when $g = 0$, $h = 5000$m, $a = 2$m/s, and $t_f = 200$sec. Compare the numerical and analytic solutions.