16.323 Principles of Optimal Control
Spring 2008

16.323 Handout #5
Prof. J. P. How April 10, 2008
Due: April 24, 2008

16.323 Homework Assignment #5

1. Suppose that the stock of a species of fish increases at a constant rate when they are not harvested. Fishing regulations set a maximum permitted rate for the harvesting of fish, which is proportional to the size of the stock. The system model is then that:

$$\dot{x} = x - ux \qquad 0 \leq u \leq h$$

where $x(t)$ is the normalized size of the stock of fish, ux is the harvesting rate, and $h < 1$ is the maximum proportional rate.

Suppose that initially $x = 1$ and that the terminal time T is fixed. The goal is to determine how to maximize the total number of fish caught, so the cost functional is given by

$$\min J = -\int_0^T ux dt$$

and obviously $x(T)$ is free.

(a) What is the optimal choice of u, and how does that depend on the costate?

(b) How many switches would you expect for this system?

(c) Given that it is unlikely that the optimal solution will end with $u = 0$, what is the maximum catch possible?

2. The dynamics of a reservoir system are given by the equations

$$\dot{x}_1(t) = -x_1(t) + u(t) \qquad (1)$$
$$\dot{x}_2(t) = x_1(t) \qquad (2)$$

where $x_1(t)$ and $x_2(t)$ correspond to the water height in each tank, and the inflow is constrained so that $0 \leq u(t) \leq 1$. Initially we have $x_1(0) = x_2(0) = 0$.

The objective is to maximize $x_2(1)$ subject to the constraint that $x_1(1) = 0.5$. Find the optimal input strategy $u(t)$ for the problem.

3. Consider the minimum-time optimal control problem for the following harmonic oscillator:

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 + u, \qquad |u| \leq 1 \tag{3}$$

that leads both states to the origin: $\mathbf{x}(t_f) = (0,0)$.

(a) Form the Hamiltonian for this system and find the optimal control law.

(b) Use your result in part (a) to support the following conclusions:

- The time-optimal control must be piecewise-constant and switches between ± 1.
- The time-optimal control can remain constant for no more than π units of time.
- There is no upperbound to the number of switches of the time-optimal control.
- The function $p_2(t)$ cannot be zero for a finite period of time, so there is no possibility of singular control.

(c) Given the results of (b), now let us consider the response of the system under the control action $u = \pm 1$. Using the fact that the state equation can be manipulated to the form:

$$(x_1 - u)\dot{x}_1 + x_2\dot{x}_2 = 0, \tag{4}$$

show that each state can be expressed as

$$x_1 - u = c\cos\theta, \qquad x_2 = c\sin\theta \tag{5}$$

for fixed u (Hint: Eq. 4 can be regarded as the time derivative of some $A(\mathbf{x},u)^2 + B(\mathbf{x},u)^2 = R^2$ expression.)

(d) Use the result in (c) to show that the response curves are circles in the x_1-x_2 plane, so that if $u = 1$, they are circles about $(1,0)$, and if $u = -1$, they are circles about $(-1,0)$. Given this information, sketch what you think the time optimal response will be in the (x_1, x_2) plane given an initial point of $(1,1)$.

4. It is known that the stochastic version of the HJB equation is written as:

$$-J_t^\star = \min_{\mathbf{u}} \left\{ g(\mathbf{x},\mathbf{u},t) + J_{\mathbf{x}}^\star \mathbf{a}(\mathbf{x},\mathbf{u},t) + \frac{1}{2}\texttt{trace}\left(J_{\mathbf{xx}}^\star G(\mathbf{x},\mathbf{u},t) W G^T(\mathbf{x},\mathbf{u},t) \right) \right\} \tag{6}$$

for the optimal control with the cost function

$$J = E\left\{ m(\mathbf{x}(t_f)) + \int_t^{t_f} g(\mathbf{x}(t),\mathbf{u}(t),t)dt \;\middle|\; \mathbf{x}(t) = \text{given} \right\} \tag{7}$$

and the dynamics

$$\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x},\mathbf{u},t) + G(\mathbf{x},t)\mathbf{w}. \tag{8}$$

where $E\left[\mathbf{w}(t)\mathbf{w}^T(\tau)\right] = W\delta(t-\tau)$.

(a) In case the dynamics is LTI and the objective is quadratic as:

$$\begin{aligned} \mathbf{a}(\mathbf{x},\mathbf{u},t) &= A\mathbf{x} + B\mathbf{u}, \qquad G(\mathbf{x},t) \equiv G \qquad (9) \\ g(\mathbf{x},\mathbf{u},t) &= \mathbf{x}^T Q\mathbf{x} + \mathbf{u}^T R\mathbf{u}, \qquad m(\mathbf{x}(t_f)) = \mathbf{x}^T(t_f) M \mathbf{x}(t_f), \qquad (10) \end{aligned}$$

show that the optimal control law and the optimal cost have the form of

$$\begin{aligned} \mathbf{u}^\star(t) &= -R^{-1}B^T P(t)\mathbf{x}(t) \qquad (11) \\ J^\star &= \mathbf{x}^T P(t)\mathbf{x} + c(t) \qquad (12) \end{aligned}$$

where $P(t)$ and $c(t)$ are determined by

$$\begin{aligned} -\dot{P} &= A^T P + PA + Q - PBR^{-1}B^T P, \qquad P(t_f) = M \qquad (13) \\ -\dot{c} &= \mathtt{trace}(PGWG^T), \qquad c(t_f) = 0. \qquad (14) \end{aligned}$$

(b) Show that the optimal cost can be approximated as the following, under the assumption that t_f is sufficiently large and P converges to its steady-state value P_{ss}:

$$J^\star = \mathbf{x}^T P_{ss}\mathbf{x} + (t_f - t)\mathtt{trace}(P_{ss}GWG^T) \qquad (15)$$

(c) Show that

$$\lim_{t\to\infty} E\left[\mathbf{x}^T Q\mathbf{x} + \mathbf{u}^T R\mathbf{u}\right] = \mathtt{trace}(P_{ss}GWG^T) \qquad (16)$$

when $\mathbf{u}$ is determined by the optimal control law.