16.323 Principles of Optimal Control
Spring 2008

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Handout #6 April 25, 2008 **Due:** May 8, 2008

16.323 Homework Assignment #6

1. Finish 4(b) and 4(c) of Homework #5.

2. (total 15pts) Find the control input $u(t)$ sequence that minimizes the cost functional

$$J = -y(t_f)$$

subject to the state constraints

$$\dot{y}(t) = y(t)u(t) - y(t) - u^2(t)$$

for an initial condition $y(0) = 2$ and final time $t_f = 5$. Give both the control, the state response, and the costate.

3. Given the plant dynamics,

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= x_1(t) + u(t) + w(t) \\ y(t) &= x_2(t) + v(t) \end{aligned}$$

and cost function,

$$J = E\left\{\frac{1}{2}\int_0^{t_f} (3x_1^2(t) + 3x_2^2(t) + u^2(t))dt\right\}$$

where $w(t) \sim \mathcal{N}(0, 4)$ and $v(t) \sim \mathcal{N}(0, 0.5)$ are Gaussian, white noises and $t_f = 15$.

(a) Numerically integrate the Riccati equations for LQR and the LQE to find the time-varying regulator and estimator gains.

(b) The full stochastic linear optimal output feedback problem involves using $u(t) = -K\hat{\mathbf{x}}(\mathbf{t})$. For this control policy, the compensator is of the form,

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t) \quad (1)$$
$$u(t) = -C_c x_c(t) \quad (2)$$

For this example, write down the dynamic equations for the combined plant and compensator system $\begin{bmatrix} x \\ x_c \end{bmatrix}$. Simulate the full closed-loop system in MATLAB using the gains found in Part (a) and the initial conditions, $x_0 = \begin{bmatrix} 10 \\ -10 \end{bmatrix}$ and $\hat{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(c) Show that the eigenvalues of the closed-loop dynamics are equal to those of the steady-state regulator and estimator. Compare the performance of the compensator in Part (b) to the performance you obtain when you use these steady state values of the regulator and estimator gains.

(d) Find the steady-state mean-squared values of the regulator and estimator ($x_1(t)$, $x_2(t)$, $\hat{x}_1(t)$ and $\hat{x}_2(t)$).

4. Given a system with a state space representation,

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_u u(t) \qquad (3) \\ y(t) &= C_y x(t) \qquad (4) \end{aligned}$$

we have investigated various types of observer-based controllers that can be written as

$$\begin{aligned} \dot{z}(t) &= Az(t) + B_u u(t) + Hr(t) \qquad (5) \\ u(t) &= -Gz(t) \qquad (6) \end{aligned}$$

where $r(t)$ is the residual signal $[r(t) = y(t) - C_y z(t)]$ and the state vectors x and z have the same size. Assume that we have selected H and G so that the closed-loop system is stable. We can modify the original compensator in Eqs. 5–6 to obtain a very general form for the compensator:

$$\begin{aligned} \dot{z}(t) &= Az(t) + B_u u(t) + Hr(t) \quad \text{Original Comp dynamics} \qquad (7) \\ \dot{w}(t) &= Fw(t) + Qr(t) \quad \text{Other stable dynamics} \qquad (8) \\ u(t) &= -Gz(t) + Pw(t) \qquad (9) \end{aligned}$$

so that the residual is filtered by another set of stable dynamics and then added to the control signal. The dimension (k) of w is arbitrary, but the eigenvalues of F are assumed to have negative real parts.

(a) (50%) Show that a "separation" type principle still holds for the closed-loop eigenvalues.

(b) (50%) Show that under the assumptions given, this new compensator still leads to a stable closed-loop system.

Point of Interest Later control courses will show how to use this extra freedom in the compensator to achieve additional desirable properties of the closed-loop system.

5. For the following system, design a $\mathcal{H}_\infty$ controller that minimizes the cost function

$$\left\| \begin{bmatrix} W_s S \\ W_u G_c S \end{bmatrix} \right\|_\infty$$

where

$$W_s(s) = \frac{s/M + \omega_B}{s + \omega_B A}$$

with $M = 2$, $\omega_B = 5$, and $A = 0.01$. As discussed in class, the weight W_u should be adjusted so that we meet the sensitivity specification.

You should then design a $\mathcal{H}_2$ optimal (LQG) controller that yields a similar settling time of the step response and directly compare the graphs listed below for the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ controllers. One way to do this for a SISO system is set $B_u = B_w$, $C_z = C_y$, choose $R_{\text{ww}} = 0.1$ and $R_{\text{zz}} = 1$. Then set $R_{\text{uu}} = R_{\text{vv}} = \rho$ and use $\rho \ll 1$ as the basic design parameter. You can then iterate on the choice of R_{ww} and R_{zz} as needed.

Use this procedure to design a controller (follow the code given in the notes) for the following system ($\omega_1 = 1$, $\omega_2 = 5$, $\omega_3 = 10$)

$$G_1(s) = \frac{1}{(-s/\omega_1 + 1)(s/\omega_2 + 1)(s/\omega_3 + 1)}$$

For each case, draw the following plots:

- Sensitivity and inverse of W_s weight to show that you meet the specification.
- The Nyquist/Nichols plot to show that you have the correct number of encirclements
- A Bode graph that contains $G_i(s)$, $G_c(s)$, and $L = G_i G_c(s)$. Clearly label each curve and identify the gain/phase cross-over points.
- The root locus obtained by freezing the controller dynamics and then using a controller $\alpha G_c(s)$, where $\alpha \in \{0, 2\}$ (assuming that nominally $\alpha = 1$) to investigate the robustness of your design to gain errors.
- The step response of the closed-loop system.

The reason for drawing these plots is to develop some insight into what a typically $\mathcal{H}_\infty$ controller "looks like". So use your results to comment on how the $\mathcal{H}_\infty$ controller modifies the plant dynamics to achieve the desired performance.