16.333 Lecture # 10

State Space Control

- Basic state space control approaches

State Space Basics

- State space models are of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

with associated transfer function

$$G(s) = C(sI - A)^{-1}B + D$$

Note: must form symbolic inverse of matrix $(sI - A)$, which is hard.

- **Time response:** Homogeneous part $\dot{x} = Ax, \quad x(0)$ known
  - Take Laplace transform

$$X(s) = (sI - A)^{-1}x(0) \quad \Rightarrow x(t) = \mathcal{L}^{-1}\left[(sI - A)^{-1}\right]x(0)$$

  - But can show $(sI - A)^{-1} = \frac{I}{s} + \frac{A}{s^2} + \frac{A^2}{s^3} + \ldots$

    so $\mathcal{L}^{-1}\left[(sI - A)^{-1}\right] = I + At + \frac{1}{2!}(At)^2 + \ldots = e^{At}$
  - Gives $x(t) = e^{At}x(0)$ where e^{At} is *Matrix Exponential*
    - ◇ Calculate in MATLAB® using `expm.m` and not `exp.m` [1]

- **Time response:** Forced Solution – Matrix case $\dot{x} = Ax + Bu$ where x is an n-vector and u is a m-vector. Cam show

$$\begin{aligned} x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \\ y(t) &= Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t) \end{aligned}$$

  - $Ce^{At}x(0)$ is the initial response
  - $Ce^{A(t)}B$ is the impulse response of the system.

[1] MATLAB® is a trademark of the Mathworks Inc.

Dynamic Interpretation

- Since $A = T\Lambda T^{-1}$, then

$$e^{At} = Te^{\Lambda t}T^{-1} = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} - & w_1^T & - \\ & \vdots & \\ - & w_n^T & - \end{bmatrix}$$

where we have written

$$T^{-1} = \begin{bmatrix} - & w_1^T & - \\ & \vdots & \\ - & w_n^T & - \end{bmatrix}$$

which is a column of rows.

- Multiply this expression out and we get that

$$e^{At} = \sum_{i=1}^{n} e^{\lambda_i t} v_i w_i^T$$

- Assume A diagonalizable, then $\dot{x} = Ax$, $x(0)$ given, has solution

$$\begin{aligned} x(t) &= e^{At}x(0) = Te^{\Lambda t}T^{-1}x(0) \\ &= \sum_{i=1}^{n} e^{\lambda_i t} v_i \{w_i^T x(0)\} \\ &= \sum_{i=1}^{n} e^{\lambda_i t} v_i \beta_i \end{aligned}$$

- State solution is a **linear combination** of the system modes $v_i e^{\lambda_i}$

$e^{\lambda_i t}$ – Determines the **nature** of the time response

v_i – Determines extent to which each state **contributes** to that mode

β_i – Determines extent to which the initial condition excites the mode

- Note that the v_i give the relative sizing of the response of each part of the state vector to the response.

$$v_1(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t} \quad \text{mode 1}$$

$$v_2(t) = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} e^{-3t} \quad \text{mode 2}$$

- Clearly $e^{\lambda_i t}$ gives the time modulation
  - λ_i real – growing/decaying exponential response
  - λ_i complex – growing/decaying exponential damped sinusoidal

- **Bottom line:** The locations of the eigenvalues determine the pole locations for the system, thus:
  - They determine the stability and/or performance & transient behavior of the system.
  - **It is their locations that we will want to modify with the controllers.**

Full-state Feedback Controller

- Assume that the single-input system dynamics are given by

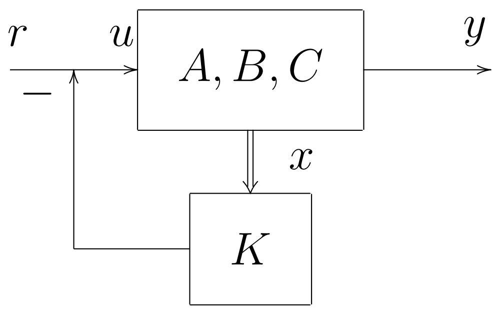
$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

so that $D = 0$.

  – The multi-actuator case is quite a bit more complicated as we would have many extra degrees of freedom.

- Recall that the system poles are given by the eigenvalues of A.

  – Want to use the input $u(t)$ to modify the eigenvalues of A to change the system dynamics.



- Assume a full-state feedback of the form:

$$u = r - Kx$$

where r is some **reference input** and the **gain** K is $\mathcal{R}^{1\times n}$

  – If $r = 0$, we call this controller a **regulator**

- Find the closed-loop dynamics:

$$\begin{aligned}\dot{x} &= Ax + B(r - Kx) \\ &= (A - BK)x + Br \\ &= A_{cl}x + Br \\ y &= Cx\end{aligned}$$

- **Objective:** Pick K so that A_{cl} has the desired properties, *e.g.*,
  - A unstable, want A_{cl} stable
  - Put 2 poles at $-2 \pm 2j$
- Note that there are n parameters in K and n eigenvalues in A, so it looks promising, but what can we achieve?
- **Example #1:** Consider:

$$\dot{x} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

  - Then

$$\det(sI - A) = (s-1)(s-2) - 1 = s^2 - 3s + 1 = 0$$

    so the system is unstable.
  - Define $u = -\begin{bmatrix} k_1 & k_2 \end{bmatrix} x = -Kx$, then

$$A_{cl} = A - BK = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 1-k_1 & 1-k_2 \\ 1 & 2 \end{bmatrix}$$

  - So then we have that

$$\det(sI - A_{cl}) = s^2 + (k_1 - 3)s + (1 - 2k_1 + k_2) = 0$$

- Thus, by choosing k_1 and k_2, we can put $\lambda_i(A_{cl})$ anywhere in the complex plane (assuming complex conjugate pairs of poles).

- To put the poles at $s = -5, \ -6$, compare the *desired characteristic equation*
$$(s+5)(s+6) = s^2 + 11s + 30 = 0$$
with the closed-loop one
$$s^2 + (k_1 - 3)x + (1 - 2k_1 + k_2) = 0$$
to conclude that
$$\left.\begin{array}{r} k_1 - 3 = 11 \\ 1 - 2k_1 + k_2 = 30 \end{array}\right\} \quad \begin{array}{l} k_1 = 14 \\ k_2 = 57 \end{array}$$
so that $K = \left[\ 14 \ \ 57\ \right]$, which is called **Pole Placement**.

- Of course, it is not always this easy, as the issue of **controllability** must be addressed.

- **Example #2:** Consider this system:
$$\dot{x} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
with the same control approach
$$A_{cl} = A - BK = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left[\ k_1 \ \ k_2\ \right] = \begin{bmatrix} 1 - k_1 & 1 - k_2 \\ 0 & 2 \end{bmatrix}$$
so that $\det(sI - A_{cl}) = (s - 1 + k_1)(s - 2) = 0$

  The feedback control can modify the pole at $s = 1$, but it cannot move the pole at $s = 2$.

- **This system cannot be stabilized with full-state feedback control.**

- What is the reason for this problem?

– It is associated with loss of controllability of the e^{2t} mode.

- Basic test for controllability: $\texttt{rank}\ \mathcal{M}_c = n$

$$\mathcal{M}_c = \left[\ B \,\middle|\, AB\ \right] = \left[\begin{bmatrix} 1 \\ 0 \end{bmatrix} \middle| \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right]$$

So that $\texttt{rank}\ \mathcal{M}_c = 1 < 2$.

- **Must assume that the pair $(A,\ B)$ are controllable.**

Ackermann's Formula

- The previous outlined a design procedure and showed how to do it by hand for second-order systems.
  - Extends to higher order (controllable) systems, but tedious.

- **Ackermann's Formula** gives us a method of doing this entire design process is one easy step.

$$K = \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix} \mathcal{M}_c^{-1} \Phi_d(A)$$

  - $\mathcal{M}_c = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}$
  - $\Phi_d(s)$ is the characteristic equation for the closed-loop poles, which we then evaluate for $s = A$.
  - It is explicit that the **system must be controllable** because we are inverting the controllability matrix.

- Revisit **Example #1:** $\Phi_d(s) = s^2 + 11s + 30$

$$\mathcal{M}_c = \left[\, B \,\middle|\, AB \,\right] = \left[\begin{bmatrix} 1 \\ 0 \end{bmatrix} \middle| \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

So

$$\begin{aligned} K &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \left(\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^2 + 11 \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} + 30I \right) \\ &= \begin{bmatrix} 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 43 & 14 \\ 14 & 57 \end{bmatrix} \right) = \begin{bmatrix} 14 & 57 \end{bmatrix} \end{aligned}$$

- Automated in Matlab: `place.m` & `acker.m` (see `polyvalm.m` too)

- Origins? For simplicity, consider a third-order system (case #2), but this extends to any order.

$$A = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}$$

  – This form is useful because the characteristic equation for the system is obvious $\Rightarrow \det(sI - A) = s^3 + a_1 s^2 + a_2 s + a_3 = 0$

- Can show that

$$\begin{aligned} A_{cl} = A - BK &= \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \\ &= \begin{bmatrix} -a_1 - k_1 & -a_2 - k_2 & -a_3 - k_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{aligned}$$

  so that the characteristic equation for the system is still obvious:

$$\Phi_{cl}(s) = \det(sI - A_{cl}) = s^3 + (a_1 + k_1)s^2 + (a_2 + k_2)s + (a_3 + k_3) = 0$$

- We then compare this with the desired characteristic equation developed from the desired closed-loop pole locations:

$$\Phi_d(s) = s^3 + (\alpha_1)s^2 + (\alpha_2)s + (\alpha_3) = 0$$

  to get that

$$\left. \begin{array}{c} a_1 + k_1 = \alpha_1 \\ \vdots \\ a_n + k_n = \alpha_n \end{array} \right\} \begin{array}{c} k_1 = \alpha_1 - a_1 \\ \vdots \\ k_n = \alpha_n - a_n \end{array}$$

- Pole placement is a very powerful tool and we will be using it for most of our state space work.

Aircraft State Space Control

- Can now design a full state feedback controller for the dynamics:

$$\dot{x}_{sp} = A_{sp}x_{sp} + B_{sp}\delta_e$$

with desired poles being at $\omega_n = 3$ and $\zeta = 0.6 \Rightarrow s = -1.8 \pm 2.4\mathbf{i}$

$$\phi_d(s) = s^2 + 3.6s + 9$$

```
Ksp=place(Asp,Bsp,[roots([1 2*0.6*3 3^2])'])
```

- Design controller $u = \begin{bmatrix} -0.0264 & -2.3463 \end{bmatrix} \begin{bmatrix} w \\ q \end{bmatrix}$

- With full model, could arrange it so phugoid poles remain in the same place, just move the ones associated with the short period mode

$$s = -1.8 \pm 2.4\mathbf{i}, \quad -0.0033 \pm 0.0672\mathbf{i}$$

```
ev=eig(A);
% damp short period, but leave the phugoid where it is
Plist=[roots([1 2*.6*3 3^2])' ev([3 4],1)'];
K1=place(A,B(:,1),Plist)
```

$$\Rightarrow u = \begin{bmatrix} 0.0026 & -0.0265 & -2.3428 & 0.0363 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix}$$

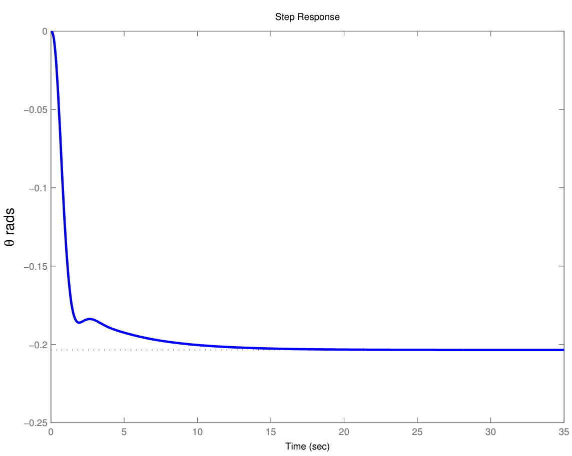
- Can also add the lag dynamics to short period model with θ included

$$
\begin{aligned}
\dot{x}_{sp} &= \tilde{A}_{sp} x_{sp} + \tilde{B}_{sp} \delta_e^a; \quad \delta_e^a = \frac{4}{s+4} \delta_e^c \\
\rightarrow \dot{x}_\delta &= -4 x_\delta + 4 \delta_e^c, \qquad \delta_e^a = x_\delta \\
\Rightarrow \begin{bmatrix} \dot{x}_{sp} \\ \dot{x}_\delta \end{bmatrix} &= \begin{bmatrix} \tilde{A}_{sp} & \tilde{B}_{sp} \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x_{sp} \\ x_\delta \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \end{bmatrix} \delta_e^c
\end{aligned}
$$

- Add $s = -3$ to desired pole list

```
Plist=[roots([1 2*.6*3 3^2])',-.25,-3];
At2=[Asp2 Bsp2(:,1);zeros(1,3) -4];Bt2=[zeros(3,1);4];
Kt=place(At2,Bt2,Plist);
step(ss(At2-Bt2*Kt2,Bt2,[0 0 1 0],0),35)
```

$$
u = \begin{bmatrix} 0.0011 & -3.4617 & -4.9124 & 0.5273 \end{bmatrix} \begin{bmatrix} w \\ q \\ \theta \\ x_\delta \end{bmatrix}
$$



- No problem working with larger systems with state space tools
- Main control issue is finding "good" locations for closed-loop poles

Estimators/Observers

- **Problem:** So far we have assumed that we have full access to the state $x(t)$ when we designed our controllers.
  - Most often all of this information is not available.

- Usually can only feedback information that is developed from the sensors measurements.
  - Could try "output feedback"

$$u = Kx \quad \Rightarrow \quad u = \hat{K}y$$

  - Same as the proportional feedback we looked at at the beginning of the root locus work.
  - This type of control is very difficult to design in general.

- **Alternative approach:** Develop a replica of the dynamic system that provides an "estimate" of the system states based on the measured output of the system.

- **New plan:**
  1. Develop estimate of $x(t)$ that will be called $\hat{x}(t)$.
  2. Then switch from $u = -Kx(t)$ to $u = -K\hat{x}(t)$.

- Two key questions:
  - How do we find $\hat{x}(t)$?
  - Will this new plan work?

Estimation Schemes

- Assume that the system model is of the form:

$$\begin{aligned} \dot{x} &= Ax + Bu \ , \quad x(0) \text{ unknown} \\ y &= Cx \end{aligned}$$

where

1. A, B, and C are known.
2. $u(t)$ is known
3. Measurable outputs are $y(t)$ from $C \neq I$

- **Goal:** Develop a dynamic system whose state

$$\hat{x}(t) = x(t)$$

for all time $t \geq 0$. Two primary approaches:

– Open-loop.

– Closed-loop.

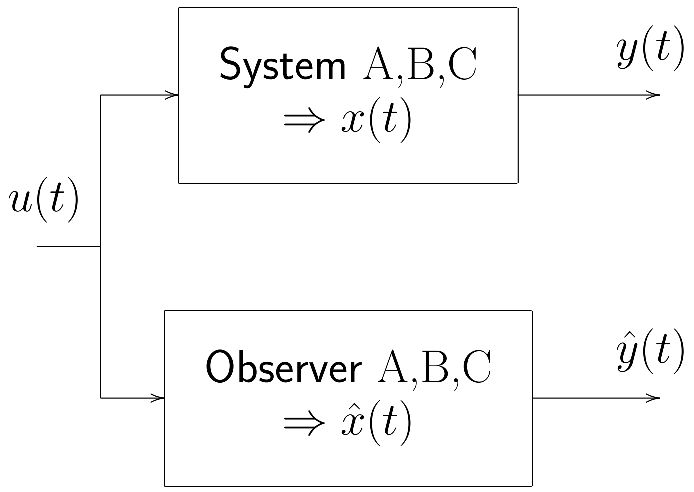
Open-loop Estimator

- Given that we know the plant matrices and the inputs, we can just perform a simulation that runs in parallel with the system

$$\dot{\hat{x}}(t) = A\hat{x} + Bu(t)$$

- Then $\hat{x}(t) \equiv x(t)\ \forall\ t$ provided that $\hat{x}(0) = x(0)$

- **Major Problem:** We do not know $x(0)$



- Analysis of this case. Start with:

$$\begin{aligned} \dot{x}(t) &= Ax + Bu(t) \\ \dot{\hat{x}}(t) &= A\hat{x} + Bu(t) \end{aligned}$$

- Define the **estimation error**: $\tilde{x}(t) = x(t) - \hat{x}(t)$.
  - Now want $\tilde{x}(t) = 0\ \forall\ t$.
  - But is this realistic?

- Subtract to get:

$$\frac{d}{dt}(x - \hat{x}) = A(x - \hat{x}) \quad \Rightarrow \quad \dot{\tilde{x}}(t) = A\tilde{x}$$

which has the solution

$$\tilde{x}(t) = e^{At}\tilde{x}(0)$$

  – Gives the estimation error in terms of the initial error.

- Does this guarantee that $\tilde{x} = 0 \ \forall \ t$?
Or even that $\tilde{x} \rightarrow 0$ as $t \rightarrow \infty$? (which is a more realistic goal).

  – Response is fine if $\tilde{x}(0) = 0$. But what if $\tilde{x}(0) \neq 0$?

- If A stable, then $\tilde{x} \rightarrow 0$ as $t \rightarrow \infty$, but the dynamics of the estimation error are completely determined by the open-loop dynamics of the system (eigenvalues of A).

  – Could be very slow.

  – No obvious way to modify the estimation error dynamics.

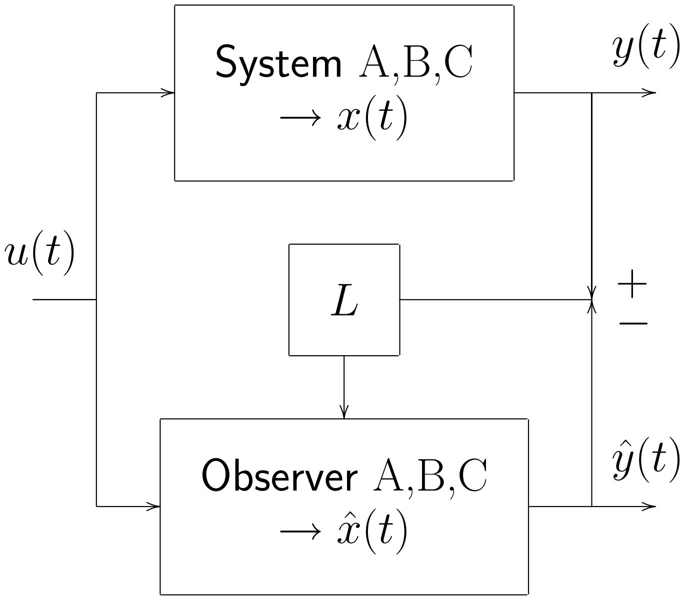
- Open-loop estimation **does not seem to be a very good idea.**

Closed-loop Estimator

- Obvious way to fix the problem is to use the additional information available:
  - How well does the estimated output match the measured output?

$$\text{Compare: } \; y = Cx \quad \text{with} \quad \hat{y} = C\hat{x}$$

  - Then form $\tilde{y} = y - \hat{y} \equiv C\tilde{x}$



- **Approach:** Feedback $\tilde{y}$ to improve our estimate of the state. Basic form of the estimator is:

$$\begin{aligned}
\dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + \boxed{L\tilde{y}(t)} \\
\hat{y}(t) &= C\hat{x}(t)
\end{aligned}$$

  where L is a **user selectable gain matrix.**

- **Analysis:**

$$\begin{aligned}
\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}} &= [Ax + Bu] - [A\hat{x} + Bu + L(y - \hat{y})] \\
&= A(x - \hat{x}) - L(Cx - C\hat{x}) \\
&= A\tilde{x} - LC\tilde{x} = (A - LC)\tilde{x}
\end{aligned}$$

- So the closed-loop estimation error dynamics are now

$$\dot{\tilde{x}} = (A - LC)\tilde{x} \;\text{ with solution }\; \tilde{x}(t) = e^{(A-LC)t}\,\tilde{x}(0)$$

- **Bottom line:** Can select the gain L to attempt to improve the convergence of the estimation error (and/or speed it up).
  - But now must worry about observability of the system model.

- Note the similarity:

  - **Regulator Problem:** pick K for $A - BK$
    - ◇ Choose $K \in \mathcal{R}^{1\times n}$ (SISO) such that the closed-loop poles

$$\det(sI - A + BK) = \Phi_c(s)$$

      are in the desired locations.
  - **Estimator Problem:** pick L for $A - LC$
    - ◇ Choose $L \in \mathcal{R}^{n\times 1}$ (SISO) such that the closed-loop poles

$$\det(sI - A + LC) = \Phi_o(s)$$

      are in the desired locations.

- These problems are obviously very similar – in fact they are called **dual problems**.

Estimation Gain Selection

- For regulation, were concerned with controllability of (A, B)

  For a controllable system we can place the eigenvalues of $A - BK$ arbitrarily.

- For estimation, were concerned with observability of pair (A, C).

  For an observable system we can place the eigenvalues of $A - LC$ arbitrarily.

- Test using the observability matrix:

$$\texttt{rank}\ \mathcal{M}_o \triangleq \texttt{rank} \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$$

- The procedure for selecting L is very similar to that used for the regulator design process.

- One approach:
  - Note that the poles of $(A - LC)$ and $(A - LC)^T$ are identical.
  - Also we have that $(A - LC)^T = A^T - C^T L^T$
  - So designing L^T for this transposed system looks like a standard regulator problem $(A - BK)$ where

$$\begin{aligned} A &\Rightarrow A^T \\ B &\Rightarrow C^T \\ K &\Rightarrow L^T \end{aligned}$$

So we can use

$$K_e = \texttt{acker}(A^T, C^T, P)\ , \quad L \equiv K_e^T$$

- Note that the estimator equivalent of Ackermann's formula is that

$$L = \Phi_e(s)\mathcal{M}_o^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Simple Estimator Example

- Simple system

$$
\begin{aligned}
A &= \begin{bmatrix} -1 & 1.5 \\ 1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad x(0) = \begin{bmatrix} -0.5 \\ -1 \end{bmatrix} \\
C &= \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0
\end{aligned}
$$

  – Assume that the initial conditions are not well known.

  – System stable, but $\lambda_{\max}(A) = -0.18$

  – Test observability:

$$
\texttt{rank} \begin{bmatrix} C \\ CA \end{bmatrix} = \texttt{rank} \begin{bmatrix} 1 & 0 \\ -1 & 1.5 \end{bmatrix}
$$

- Use open and closed-loop estimators. Since the initial conditions are not well known, use $\hat{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

- Open-loop estimator:

$$
\begin{aligned}
\dot{\hat{x}} &= A\hat{x} + Bu \\
\hat{y} &= C\hat{x}
\end{aligned}
$$

- Closed-loop estimator:

$$
\begin{aligned}
\dot{\hat{x}} &= A\hat{x} + Bu + L\tilde{y} = A\hat{x} + Bu + L(y - \hat{y}) \\
&= (A - LC)\hat{x} + Bu + Ly \\
\hat{y} &= C\hat{x}
\end{aligned}
$$

  – Which is a dynamic system with poles given by $\lambda_i(A - LC)$ and which takes the measured plant outputs as an input and generates an estimate of x.

- Typically simulate both systems together for simplicity

- Open-loop case:

$$
\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx \\
\dot{\hat{x}} &= A\hat{x} + Bu \\
\hat{y} &= C\hat{x}
\end{aligned}
$$

$$
\Rightarrow \begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} u \ , \quad \begin{bmatrix} x(0) \\ \hat{x}(0) \end{bmatrix} = \begin{bmatrix} -0.5 \\ -1 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\begin{bmatrix} y \\ \hat{y} \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}
$$

- Closed-loop case:

$$
\begin{aligned}
\dot{x} &= Ax + Bu \\
\dot{\hat{x}} &= (A - LC)\hat{x} + Bu + LCx
\end{aligned}
$$

$$
\Rightarrow \begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & 0 \\ LC & A - LC \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} u
$$

- Example uses a strong $u(t)$ to shake things up

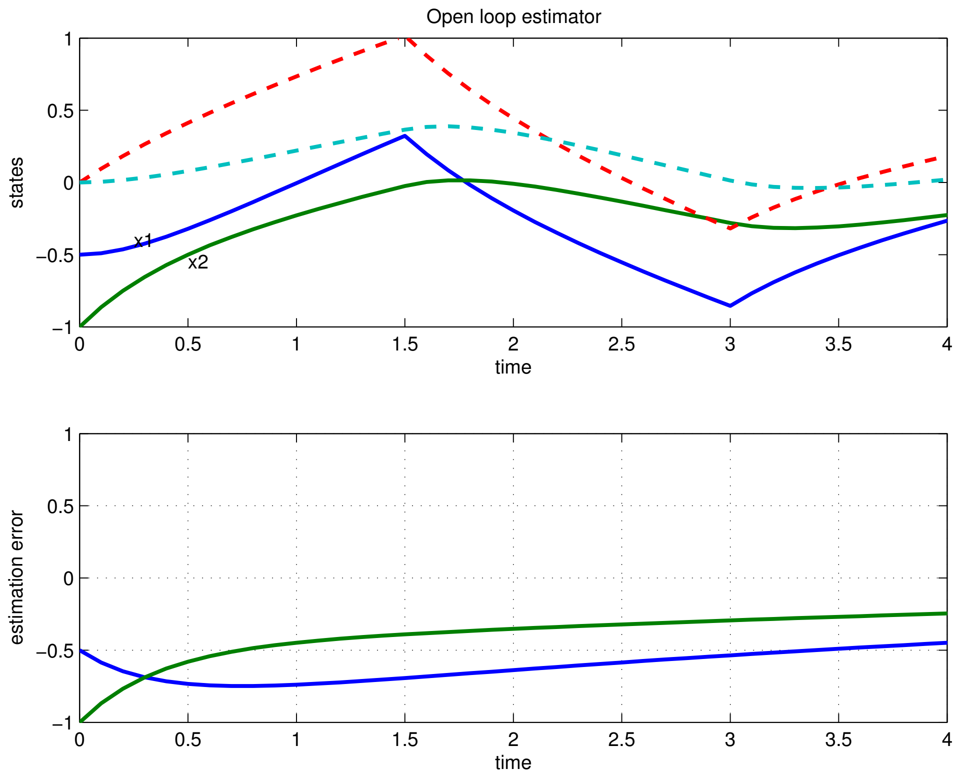


Figure 1: Open-loop estimator. Estimation error converges to zero, but very slowly.

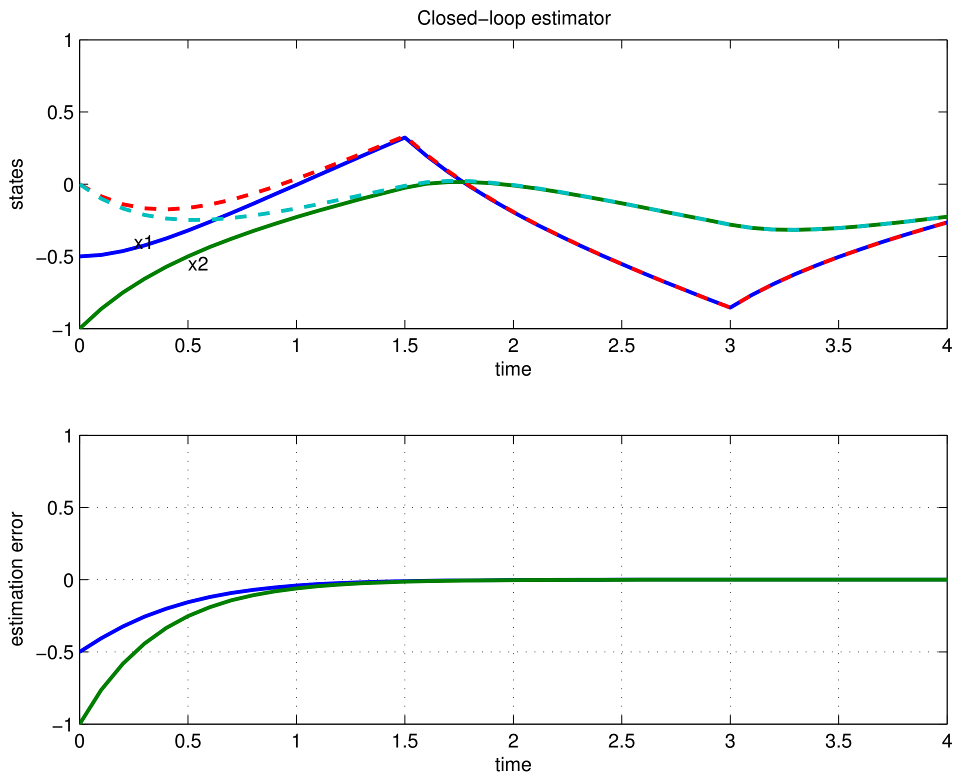


Figure 2: Closed-loop estimator. Convergence looks much better.

Aircraft Estimation Example

- Take Short period model and assume that we can measure q. Can we estimate the motion associated with the short period mode?

$$\begin{aligned} \dot{x}_{sp} &= A_{sp}x_{sp} + B_{sp}u \\ y &= \begin{bmatrix} 0 & 1 \end{bmatrix} x_{sp} \end{aligned}$$

  – Take $x_{sp}(0) = [-0.5; -0.05]^T$

- System stable, so could use an open loop estimator
- For closed-loop estimator, put desired poles at -3, -4
- For the various dynamics models as before

```
Csp=[0 1]; % sense q
Ke=place(Asp',Csp',[-3 -4]);Le=Ke';
```

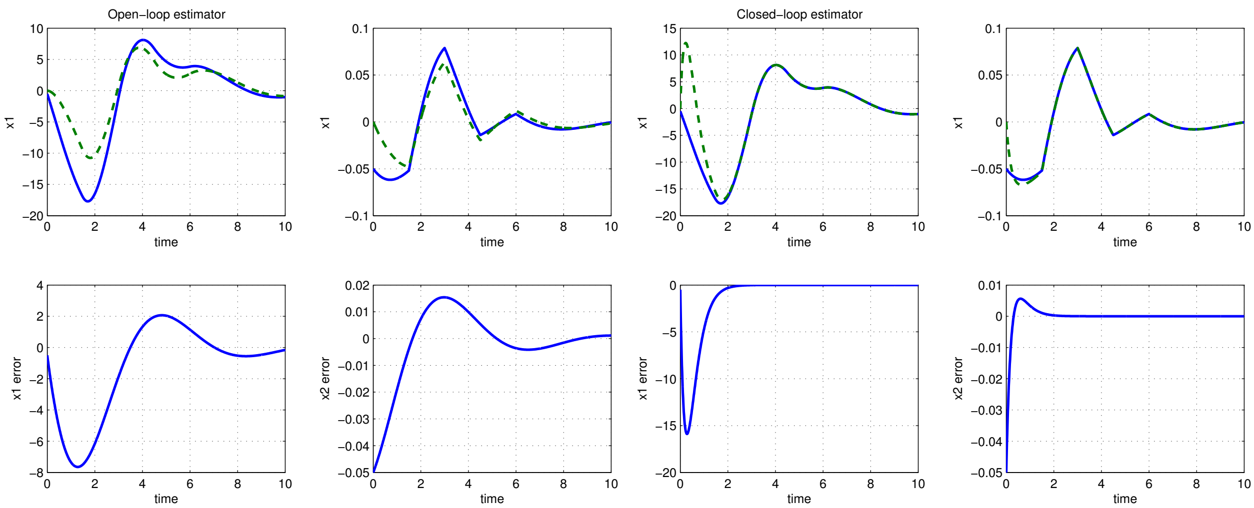


Figure 3: Closed-loop estimator. Convergence looks much better.

- As expected, the OL estimator does not do well, but the closed-loop one converges nicely

Where to put the Estimator Poles?

- Location heuristics for poles still apply – use Bessel, ITAE, ...
  - Main difference: probably want to make the estimator faster than you intend to make the regulator – should enhance the control, which is based on $\hat{x}(t)$.
  - ROT: Factor of 2–3 in the time constant $\zeta\omega_n$ associated with the regulator poles.

- **Note:** When designing a regulator, were concerned with "bandwidth" of the control getting too high $\Rightarrow$ often results in control commands that *saturate* the actuators and/or change rapidly.

- Different concerns for the estimator:
  - Loop closed inside computer, so saturation not a problem.
  - However, the measurements y are often "noisy", and we need to be careful how we use them to develop our state estimates.

$\Rightarrow$ **High bandwidth estimators** tend to accentuate the effect of sensing noise in the estimate.

- State estimates tend to "track" the measurements, which are fluctuating randomly due to the noise.

$\Rightarrow$ **Low bandwidth estimators** have lower gains and tend to rely more heavily on the plant model

- Essentially an open-loop estimator – tends to ignore the measurements and just uses the plant model.

- Can also develop an **optimal estimator** for this type of system.
  - Which is apparently what Kalman did one evening in 1958 while taking the train from Princeton to Baltimore...
  - **Balances effect** of the various types of random noise in the system on the estimator:

$$\begin{aligned}\dot{x} &= Ax + Bu + B_w w \\ y &= Cx + v\end{aligned}$$

where:

◇ w: "process noise" – models uncertainty in the system model.

◇ v: "sensor noise" – models uncertainty in the measurements.

Final Thoughts

- Note that the feedback gain L in the estimator only stabilizes the estimation error.
  - If the system is unstable, then the state estimates will also go to ∞, with zero error from the actual states.

- Estimation is an important concept of its own.
  - Not always just "part of the control system"
  - Critical issue for guidance and navigation system

- More complete discussion requires that we study stochastic processes and optimization theory.

- **Estimation is all about which do you trust more: your measurements or your model.**

Combined Regulator and Estimator

- As advertised, we can change the previous control $u = -Kx$ to the new control $u = -K\hat{x}$ (same K). We now have

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(y - \hat{y}) \\ \hat{y} &= C\hat{x} \end{aligned}$$

with closed-loop dynamics

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & -BK \\ LC & A - BK - LC \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} \quad \Rightarrow \quad \dot{x}_{\text{cl}} = A_{\text{cl}} x_{\text{cl}}$$

- Not obvious that this system will even be stable: $\lambda_i(A_{\text{cl}}) < 0$?

- To analyze, introduce $\tilde{x} = x - \hat{x}$, and the *similarity transform*

$$T = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} = T^{-1}$$

- Rewrite the dynamics in terms of the state $\begin{bmatrix} x \\ \tilde{x} \end{bmatrix} = T \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$

$$A_{\text{cl}} \;\Rightarrow\; T^{-1} A_{\text{cl}} T \equiv \overline{A_{\text{cl}}}$$

and when you work through the math, you get

$$\overline{A_{\text{cl}}} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \quad \text{!!!}$$

- **Absolutely key points:**
  1. $\lambda_i(A_{\text{cl}}) \equiv \lambda_i(\overline{A_{\text{cl}}})$ why?
  2. $\overline{A_{\text{cl}}}$ is block upper triangular, so can find poles by inspection:

$$\det(sI - \overline{A_{\text{cl}}}) = \det(sI - (A - BK)) \cdot \det(sI - (A - LC))$$

The closed-loop poles of the system consist of the union of the regulator and estimator poles

- So we can design the estimator and regulator separately with confidence that combination of the two will work **VERY** well.

- Compensator is a combination of the estimator and regulator.

$$\begin{aligned}
\dot{\hat{x}} &= A\hat{x} + Bu + L(y - \hat{y}) \\
&= (A - BK - LC)\hat{x} + Ly \\
u &= -K\hat{x}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \dot{x}_c &= A_c x_c + B_c y \\
u &= -C_c x_c
\end{aligned}$$

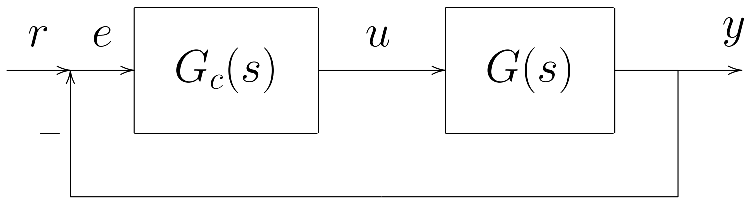
  – Keep track of this minus sign. We need one in the feedback path, but we can move it around to suit our needs.

- Let $G_c(s)$ be the compensator transfer function where

$$\begin{aligned}\frac{u}{y} &= -C_c(sI - A_c)^{-1}B_c = -G_c(s) \\ &= -K(sI - (A - BK - LC))^{-1}L\end{aligned}$$

  so by my definition, $\Rightarrow u = -G_c y \;\equiv G_c(-y)$

- Reason for making the definition is that when we implement the controller, we often do not just feedback $-y(t)$, but instead have to include a *reference command* $r(t)$
  - Use **servo approach** and feed back $e(t) = r(t) - y(t)$ instead



  - So now $u = G_c e = G_c(r - y)$.
  - And if $r = 0$, then we still have $u = G_c(-y)$

- Important points:
  - Closed-loop system will be stable, but the compensator dynamics need not be.
  - Often very simple and useful to provide classical interpretations of the compensator dynamics $G_c(s)$.

- Mechanics of closing the loop

$$\begin{aligned} G(s) : \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

$$\begin{aligned} G_c(s) : \dot{x}_c &= A_c x_c + B_c \mathbf{e} \\ u &= C_c x_c \end{aligned}$$

and $\mathbf{e} = r - y$, $u = G_c\mathbf{e}$, $y = Gu$.

- Loop dynamics $L = GG_c \Rightarrow y = L(s)\mathbf{e}$

$$\begin{aligned} \dot{x} &= Ax + Bu = Ax + BC_c x_c \\ \dot{x}_c &= A_c x_c + B_c \mathbf{e} \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{x}_c \end{bmatrix} &= \begin{bmatrix} A & BC_c \\ 0 & A_c \end{bmatrix} \begin{bmatrix} x \\ x_c \end{bmatrix} + \begin{bmatrix} 0 \\ B_c \end{bmatrix} \mathbf{e} \\ y &= \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ x_c \end{bmatrix} \end{aligned}$$

- Now form the closed-loop dynamics by inserting $\mathbf{e} = r - y$

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{x}_c \end{bmatrix} &= \begin{bmatrix} A & BC_c \\ 0 & A_c \end{bmatrix} \begin{bmatrix} x \\ x_c \end{bmatrix} + \begin{bmatrix} 0 \\ B_c \end{bmatrix} \left(r - \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ x_c \end{bmatrix} \right) \\ &= \begin{bmatrix} A & BC_c \\ -B_c C & A_c \end{bmatrix} \begin{bmatrix} x \\ x_c \end{bmatrix} + \begin{bmatrix} 0 \\ B_c \end{bmatrix} r \\ y &= \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ x_c \end{bmatrix} \end{aligned}$$

Performance Issue

- Often find with state space controllers that the DC gain of the closed loop system is not 1. So $y \neq r$ in steady state.

- Relatively simple fix is to modify the original controller with scalar N

$$u = r - Kx \Rightarrow u = Nr - Kx$$

- Closed-loop system on page 5 becomes

$$\left.\begin{aligned} \dot{x} &= Ax + B(Nr - Kx) = A_{cl}x + BNr \\ y &= Cx \end{aligned}\right\} G_{cl}(s) = C(sI - A_{cl})^{-1}BN$$

  – Analyze steady state step response $\Rightarrow$ $y_{ss} = G_{cl}(0) r_{step}$

$$G_{cl}(0) = C(-A_{cl})^{-1}BN$$

  – And pick N so that $G_{cl}(0) = 1 \Rightarrow N = \frac{1}{(C(-A_{cl})^{-1}B)}$

- A bit more complicated with a combined estimator and regulator
  – One simple way (not the best) of achieving a similar goal is to add N to r and force $G_{cl}(0) = 1$
  – Now the closed-loop dynamics on page 29 become:

$$\left.\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{x}_c \end{bmatrix} &= A_{cl} \begin{bmatrix} x \\ x_c \end{bmatrix} + B_{cl}Nr \\ y &= C_{cl} \begin{bmatrix} x \\ x_c \end{bmatrix} \end{aligned}\right\} \rightarrow N = \frac{1}{(C_{cl}(-A_{cl})^{-1}B_{cl})}$$

- Note that this fixes the steady state tracking error problems, but in my experience can create strange transients (often NMP).

Example: Compensator Design

$$G(s) = \frac{1}{s^2+s+1} \;\Rightarrow\; \begin{array}{l} \dot{x} = Ax + Bu \\ y = Cx \end{array}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \; C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

- Regulator: Want regulator poles to have a time constant of $\tau_c = 1/(\zeta\omega_n) = 0.25$ sec $\Rightarrow \lambda(A - BK_r) = -4 \pm 4j$ which can be found using **place** or **acker**

  ```
  K_r=acker(a,b,[-4+4*j;-4-4*j]);
  ```

  to give $K_r = [\,31 \;\; 7\,]$

- Estimator: want the estimator poles to be faster, so use $\tau_e = 1/(\zeta\omega_n) = 0.1$ sec. Use real poles, $\Rightarrow \lambda(A - L_eC) = -10$

  ```
  L_e=acker(a',c',[-10 -10]')';
  ```

  which gives $L_e = \begin{bmatrix} 19 \\ 80 \end{bmatrix}$

- Form compensator $G_c(s)$

  ```
  ac=a-b*K_r-L_e*c;bc=L_e;cc=K_r;dc=0;
  ```

$$A_c = \begin{bmatrix} -19 & 1 \\ -112 & -8 \end{bmatrix} \; B_c = \begin{bmatrix} 19 \\ 80 \end{bmatrix} \; C_c = \begin{bmatrix} 31 & 7 \end{bmatrix}$$

$$G_c(s) = 1149\frac{(s+2.5553)}{s^2+27s+264} = \frac{u}{e}$$

Low frequency zero, with higher frequency poles (like a lead)

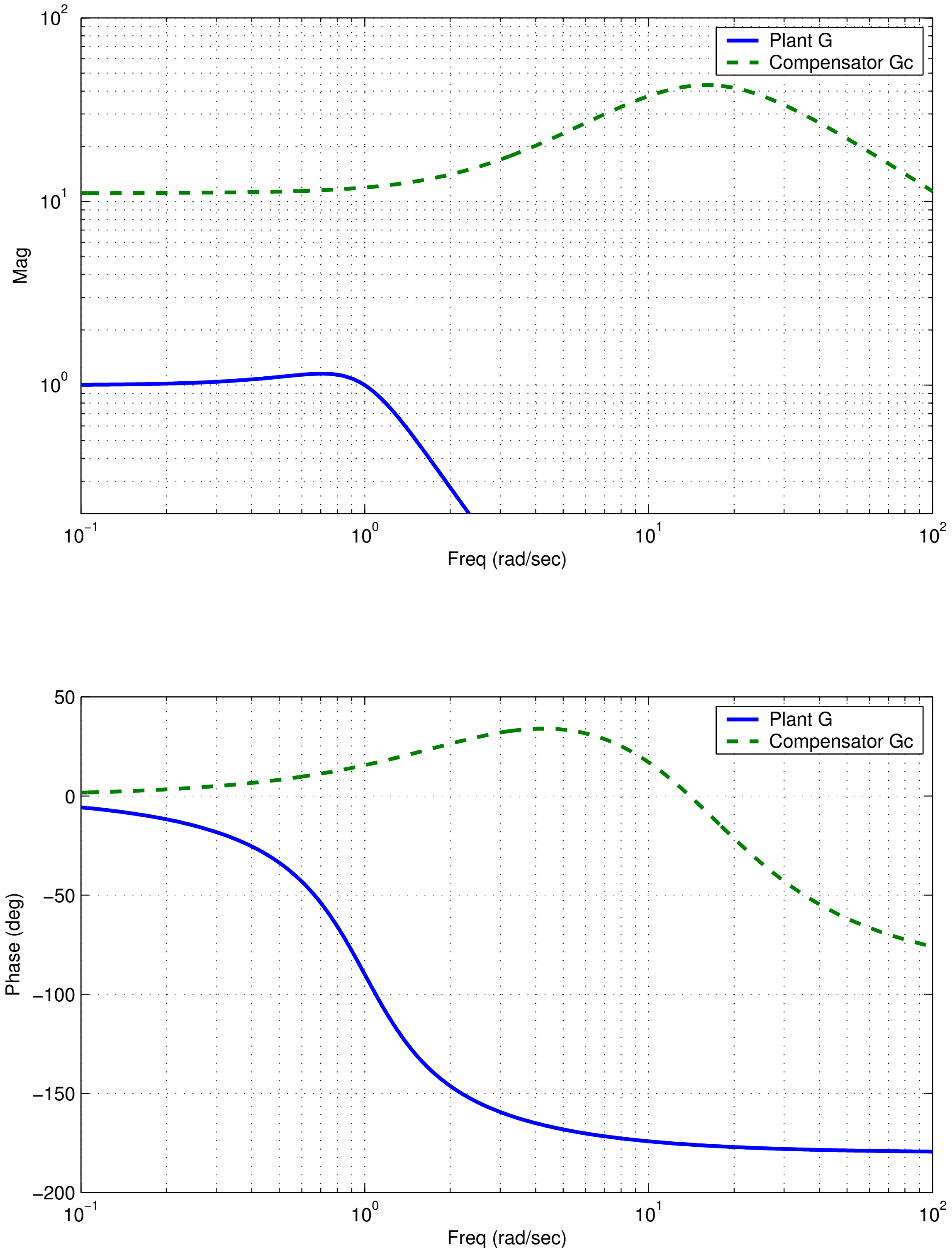


Figure 4: The compensator does indeed look like a high frequency lead (amplification from 2–16 rad/sec). Plant pretty simple looking.

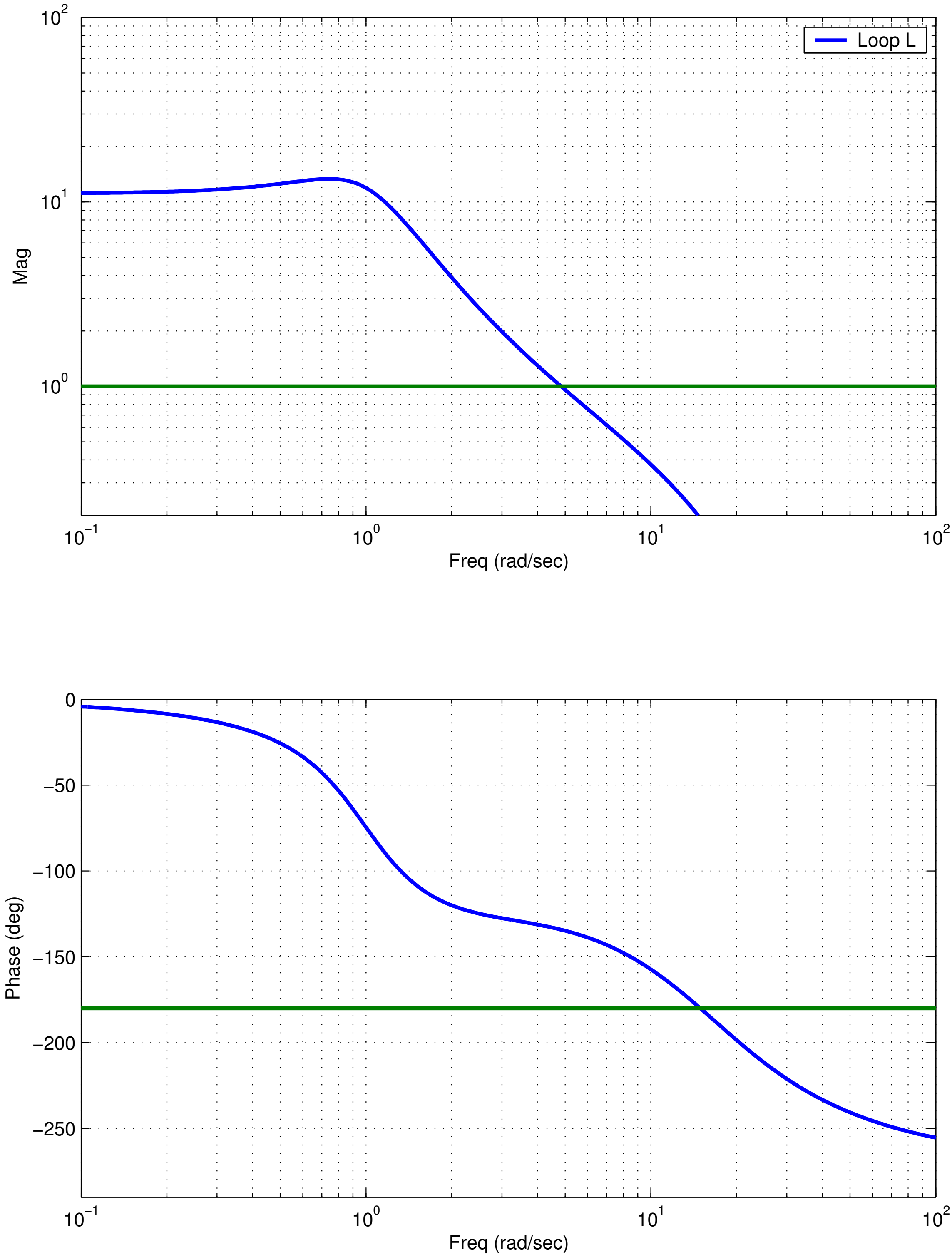


Figure 5: The loop transfer function $L = G_c G$ shows a slope change around $\omega_c = 5$ rad/sec due to the effect of the compensator. Significant gain and phase margins.

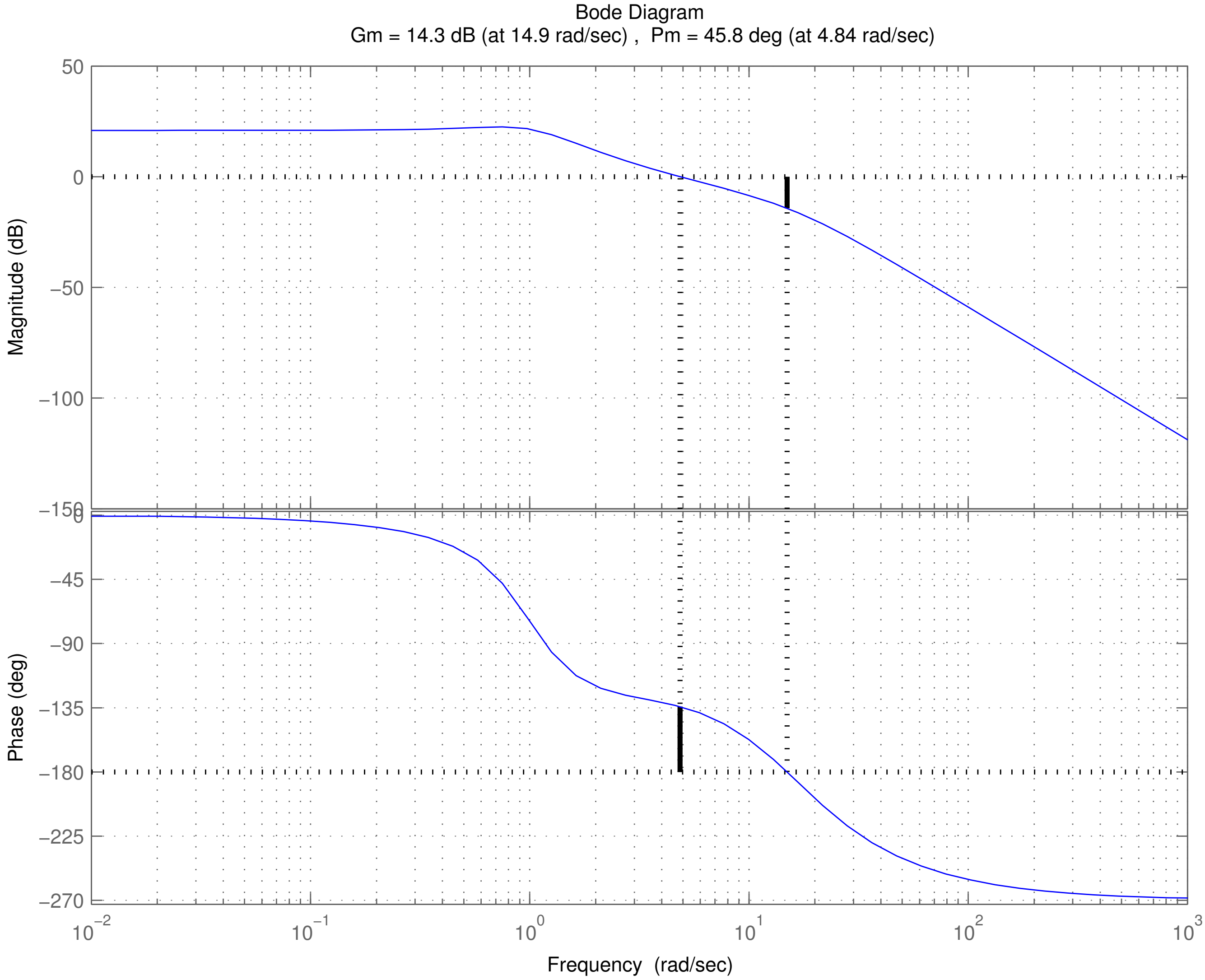


Figure 6: Quite significant gain and phase margins.

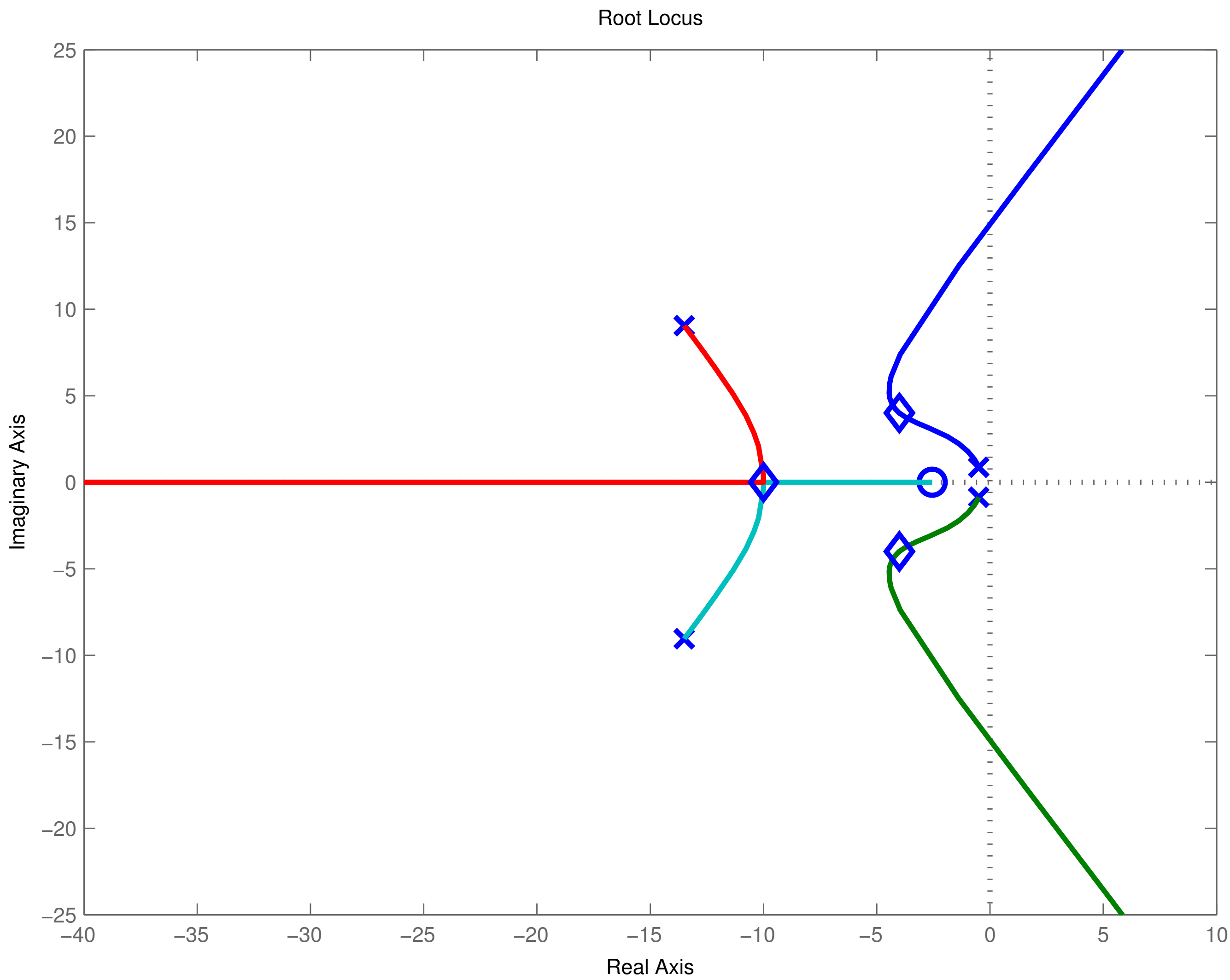


Figure 7: Freeze the compensator poles and zeros and draw a root locus versus an additional plant gain α, $G(s) \Rightarrow \tilde{G}(s) = \frac{\alpha}{(s^2+s+1)}$. Note location of the closed-loop poles!!

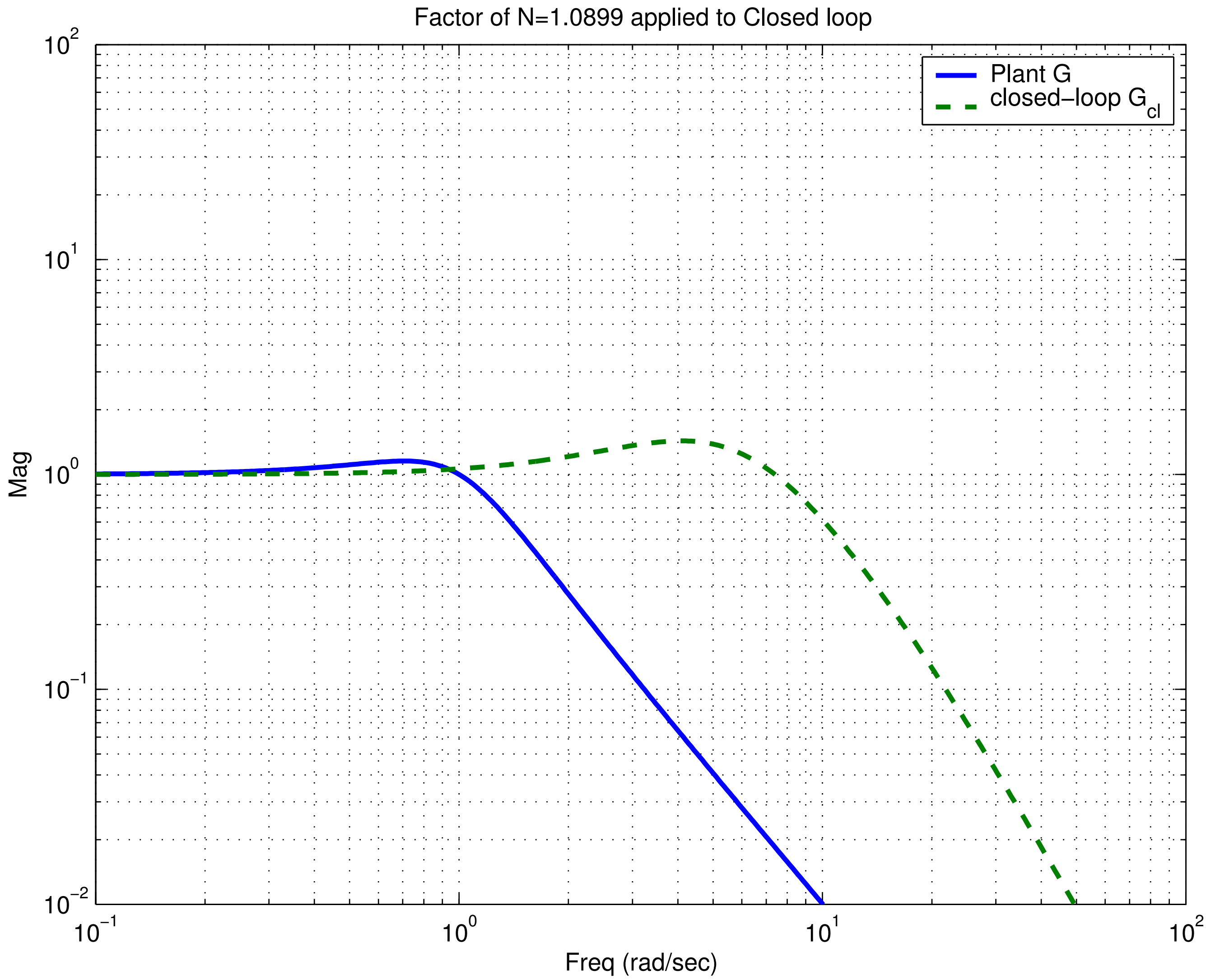


Figure 8: Closed-loop transfer – system bandwidth has increased substantially.

Estimator Design (est1.m)

```
clear all
close all
figure(1);clf
set(gcf,'DefaultLineLineWidth',2)
set(gcf,'DefaultlineMarkerSize',10)
figure(2);clf
set(gcf,'DefaultLineLineWidth',2)
set(gcf,'DefaultlineMarkerSize',10)

load b747 % get A B Asp Bsp
Csp=[0 1]; % sense q

Ke=place(Asp',Csp',[-3 -4]);Le=Ke';

xo=[-.5;-.05]; % start somewhere

t=[0:.01:10];N=floor(.15*length(t));
% hit on the system with an input
%u=0;u=[ones(15,1);-ones(15,1);ones(15,1)/2;-ones(15,1)/2;zeros(41,1)]/5;
u=0;u=[ones(N,1);-ones(N,1);ones(N,1)/2;-ones(N,1)/2]/20;
u(length(t))=0;

[y,x]=lsim(Asp,Bsp,Csp,0,u,t,xo);
plot(t,y)

% closed-loop estimator
% hook both up so that we can simulate them at the same time
% bigger state = state of the system then state of the estimator
A_cl=[Asp zeros(size(Asp));Le*Csp Asp-Le*Csp];
B_cl=[Bsp;Bsp];
C_cl=[Csp zeros(size(Csp));zeros(size(Csp)) Csp];
D_cl=zeros(2,1);

% note that we start the estimators at zero, since that is
% our current best guess of what is going on (i.e. we have no clue :-) )
%
[y_cl,x_cl]=lsim(A_cl,B_cl,C_cl,D_cl,u,t,[xo;0;0]);
figure(1)
subplot(221)
plot(t,x_cl(:,[1]),t,x_cl(:,[3]),'--')
ylabel('x1');title('Closed-loop estimator');xlabel('time');grid
subplot(222)
plot(t,x_cl(:,[2]),t,x_cl(:,[4]),'--')
ylabel('x1');xlabel('time');grid
subplot(223)
plot(t,x_cl(:,[1])-x_cl(:,[3]))
ylabel('x1 error');xlabel('time');grid
subplot(224)
plot(t,x_cl(:,[2])-x_cl(:,[4]))
ylabel('x2 error');xlabel('time');grid
print -depsc spest_cl.eps
jpdf('spest_cl')

% open-loop estimator
% hook both up so that we can simulate them at the same time
% bigger state = state of the system then state of the estimator
A_ol=[Asp zeros(size(Asp));zeros(size(Asp)) Asp];
B_ol=[Bsp;Bsp];
C_ol=[Csp zeros(size(Csp));zeros(size(Csp)) Csp];
D_ol=zeros(2,1);

[y_ol,x_ol]=lsim(A_ol,B_ol,C_ol,D_ol,u,t,[xo;0;0]);
figure(2)
subplot(221)
plot(t,x_ol(:,[1]),t,x_ol(:,[3]),'--')
ylabel('x1');title('Open-loop estimator');xlabel('time');grid
```

```
subplot(222)
plot(t,x_ol(:,[2]),t,x_ol(:,[4]),'--')
ylabel('x1');xlabel('time');grid
subplot(223)
plot(t,x_ol(:,[1])-x_ol(:,[3]))
ylabel('x1 error');xlabel('time');grid
subplot(224)
plot(t,x_ol(:,[2])-x_ol(:,[4]))
ylabel('x2 error');xlabel('time');grid
print -depsc spest_ol.eps
jpdf('spest_ol')
```

Regular/Estimator Design (reg_est.m)

```
% Combined estimator/regulator design for a simple system
% G= 1/(s^2+s+1)
%
% Jonathan How
% Fall 2004
%
close all;clear all
for ii=1:5
  figure(ii);clf;set(gcf,'DefaultLineLineWidth',2);set(gcf,'DefaultlineMarkerSize',10)
end

a=[0 1;-1 -1];b=[0 1]';c=[1 0];d=0;
k=acker(a,b,[-4+4*j;-4-4*j]);
l=acker(a',c',[-10 -10]')';
%
% For state space for G_c(s)
%
ac=a-b*k-l*c;bc=l;cc=k;dc=0;

G=ss(a,b,c,d);
Gc=ss(ac,bc,cc,dc);

f=logspace(-1,2,400);
g=freqresp(G,f*j);g=squeeze(g);
gc=freqresp(Gc,f*j);gc=squeeze(gc);

figure(1);clf
subplot(211)
loglog(f,abs(g),f,abs(gc),'--');axis([.1 1e2 .2 1e2])
xlabel('Freq (rad/sec)');ylabel('Mag')
legend('Plant G','Compensator Gc');grid
subplot(212)
semilogx(f,180/pi*angle(g),f,180/pi*angle(gc),'--');
axis([.1 1e2 -200 50])
xlabel('Freq (rad/sec)');ylabel('Phase (deg)');grid
legend('Plant G','Compensator Gc')

L=g.*gc;

figure(2);clf
subplot(211)
loglog(f,abs(L),[.1 1e2],[1 1]);axis([.1 1e2 .2 1e2])
xlabel('Freq (rad/sec)');ylabel('Mag')
legend('Loop L');
grid
subplot(212)
semilogx(f,180/pi*phase(L.'),[.1 1e2],-180*[1 1]);
axis([.1 1e2 -290 0])
xlabel('Freq (rad/sec)');ylabel('Phase (deg)');grid
%
% loop dynamics L = G Gc
%
al=[a b*cc;zeros(2) ac];
bl=[zeros(2,1);bc];
cl=[c zeros(1,2)];
dl=0;
figure(3)
rlocus(al,bl,cl,dl)
%
% closed-loop dynamics
% unity gain wrapped around loop L
%
acl=al-bl*cl;bcl=bl;ccl=cl;dcl=d;

N=inv(ccl*inv(-acl)*bcl)

```

```
hold on;plot(eig(acl),'d');hold off
grid
%
% closed-loop freq response
%
Gcl=ss(acl,bcl*N,ccl,dcl);
gcl=freqresp(Gcl,f*j);gcl=squeeze(gcl);

figure(4);clf
loglog(f,abs(g),f,abs(gcl),'--');
axis([.1 1e2 .01 1e2])
xlabel('Freq (rad/sec)');ylabel('Mag')
legend('Plant G','closed-loop G_{cl}');grid
title(['Factor of N=',num2str(N),' applied to Closed loop'])

figure(5);clf
margin(al,bl,cl,dl)

figure(1);orient tall;print -depsc reg_est1.eps
jpdf('reg_est1')
figure(2);orient tall;print -depsc reg_est2.eps
jpdf('reg_est2')
figure(3);print -depsc reg_est3.eps
jpdf('reg_est3')
figure(4);print -depsc reg_est4.eps
jpdf('reg_est4')
figure(5);print -depsc reg_est5.eps
jpdf('reg_est5')
```