

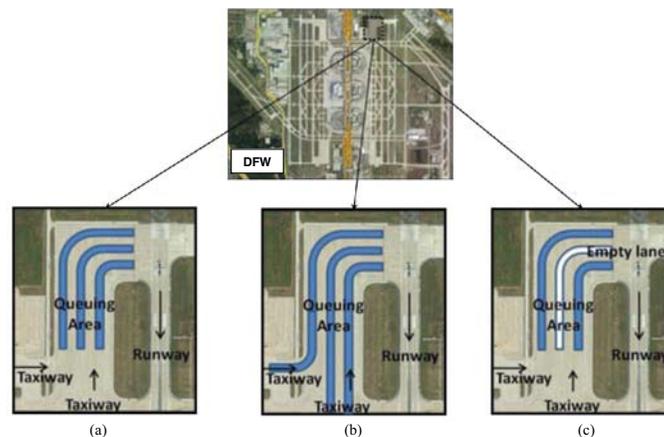
# 16.410-13 Recitation 11 Problems

## Problem 1: Mixed Integer Linear Programming

You are in charge of the scheduling of taxiways and runways for the departing aircraft in an airport. Usually, aircraft scheduling is carried out by several personnel by heuristic methods based on their experience, which requires extended training of such personnel; moreover, the resulting schedule is not guaranteed to be optimal in any sense. Your approach is to introduce Mixed-Integer Linear Programming methods to this industry for optimal scheduling of aircraft departure operations.

The figure below shows an aerial view of the DFW airport in Dallas, TX, focusing on the 17R runway, which you are in charge of. Existing heuristic commonly applied by the expert air traffic control crew is shown in the figure and listed below.

- One approach is to have a multiple queues each of which operate in a First-In-First-Out (FIFO) manner, i.e., the the first aircraft to enter the queue goes out of that queue into the runway first. The aircraft coming from the taxiways are directed to the queues arbitrarily by the scheduling crew at airport.
- The second approach is to associate each queue with a taxiway. Each aircraft enters the queue associated with the taxiway it is using to approach the runway. Each queue is operated in a FIFO manner.
- In the third approach, one queue is left empty to allow for arbitrary takeoff times for the aircraft. No FIFO queue is employed. Aircraft are held in the non-empty queueing lanes. Any aircraft that is required to depart uses the empty lane to take over other aircraft and approach the runway as long as it is empty.



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You are tasked with formulating a MILP that incorporates a similar approach, which includes several queues each of which operate in a FIFO manner. Solving this MILP will yield takeoff times for each aircraft taking into account their departure time windows as well as the queueing constraints (explained below), while maximizing certain performance metrics.

## Description:

The parameters of the problem are given as follows.

- $F$  is a set of flights (aircraft) that needs to depart.
- $P$  is a set of positions (i.e., the departure order). Note that  $|F| = |P|$ .
- $Q$  is a set of all departure queues operating in a FIFO manner.
- $a_f$  is the earliest possible departure time of flight  $f \in F$ .
- $\Delta_f$  is the departure time window of flight  $f \in F$ , i.e., flight  $f$  must depart between time  $a_f$  and  $a_f + \delta_f$ .
- $d_{i,j}$  is the minimum separation time between two flights  $i$  and  $j$  (to allow enough time for the departure of  $i$  after  $j$  leaves the runway).

You pick the following variables to formulate this problem.

- $y_{f,p}$ , where  $f \in F$  and  $p \in P$ , is a binary decision variable that takes value one whenever flight  $f$  takes departure position  $p$ .
- $x_{f,q}$ , where  $f \in F$  and  $q \in Q$ , is a binary decision variable that takes value one whenever flight  $f$  goes into queue  $q$ .
- $t_p$ , where  $p \in P$ , is a continuous decision variable that denotes the takeoff time of the flight assigned to takeoff at position  $p$ .

In this problem, you need to assign each flight to a position (through decision variables  $y_{f,p}$ ), each flight to a queue (through decision variables  $y_{f,q}$ ), and each position a departure time (through decision variables  $t_p$ ). We will assume that each aircraft arrives to the queue areas from taxiways at time  $a_j$ . Hence, the aircraft in each queue has to be ordered according to  $a_j$  of the participating aircraft. To simplify notation, you will assume that  $F$  is sorted such that for any  $i, j \in F$  with  $i > j$ , we have  $a_i > a_j$ .

## Formulation:

Formulate the following constraints in the MILP framework.

- Formulate a set of constraints that ensures each flight to have a position.

**Solution:**

$$\sum_{p \in P} y_{f,p} = 1, \quad \forall f \in F.$$

- Formulate a set of constraints that ensures each position to correspond to a flight.

**Solution:**

$$\sum_{f \in F} y_{f,p} = 1, \quad \forall p \in P.$$

- Formulate a set of constraints that assigns each flight a queue.

**Solution:**

$$\sum_{q \in Q} x_{f,q} = 1, \quad \forall f \in F.$$

- Formulate a set of constraints that ensures that each flight departs within its time window, i.e., flight  $f \in F$  departs at between  $a_f$  and  $a_f + \Delta_f$ .

**Solution:**

$$t_p \geq \sum_{f \in F} y_{f,p} a_f \quad \forall p \in P$$

$$t_p \leq \sum_{f \in F} y_{f,p} (a_f + \Delta_f) \quad \forall p \in P.$$

- Formulate a set of constraints that ensures that departures following one another are allowed enough time, i.e., if flights  $f, f' \in F$  depart at positions  $p, p-1 \in P$ , respectively, then the departure time of  $f$  must be at least  $d_{f,f'}$  later than the departure time of  $f'$ .

**Solution:**

$$t_p \geq t_{p-1} + (y_{i,p} + y_{j,p-1} - 1) d_{i,j} \quad \forall p, p-1 \in P, \quad i, j \in F$$

- Formulate a set of constraints that ensures that all queues operate in a FIFO manner, i.e., in each queue, the aircraft are ordered according to  $a_f$ .

**Solution:**

$$\sum_{u \in P} (y_{i,u}) u - \sum_{v \in P} (y_{j,v}) v \geq |P|(x_{i,q} + x_{j,q} - 2) \quad \forall q \in Q, \quad i, j \in F, \quad i > j.$$

Formulate the following objective functions in the MILP framework.

- Formulate an objective function that minimizes the total delay (i.e., the departure time minus the first possible departure time summed over all aircraft).

**Solution:**

$$\text{minimize} \quad \sum_{p \in P} t_p - \sum_{f \in F} a_f.$$

- Formulate an objective function that minimizes the maximum delay (i.e., maximum is taken over the set of all aircraft).

**Solution:** For this part we need to add in extra constraints to the problem formulation.

$$\begin{aligned} &\text{minimize} \quad T \\ &\text{subject to} \quad T \geq t_p - \sum_{f \in F} y_{f,p} a_f. \end{aligned}$$

- Formulate an objective function that minimizes the throughput (i.e., the departure time of the final flight).

**Solution:**

$$\text{minimize} \quad t_{|P|}$$

**NOTE:** This problem is based on the formulation provided in a recent paper presented at the 9th AIAA Aviation Technology, Integration, and Operations (ATIO) Conference on 21-23 Sept. 2009. The authors are with the NASA Ames Research Center. The citation goes as follows:

*G. Gupta, W. Malik, Y.C. Jung, "A Mixed Integer Linear Program for Airport Departure Scheduling", AIAA ATIO, 2009.*

## Problem 2: Probabilistic Reasoning

Recall that the fundamental axioms of probability are the following:

- For any event  $A$ ,  $P(A) \geq 0$ .
- $P(U) = 1$ .
- If  $A \cap B = \emptyset$ , then  $P(A \cup B) = P(A) + P(B)$ .

From these axioms, prove the following equality:  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ .

**Solution:** First, notice that  $A \cup B = A \cup (B \setminus A)$ . Moreover,  $A$  and  $B \setminus A$  are disjoint events. Then, using the third axiom, we can write following:

$$P(A \cup B) = P(A \cup (B \setminus A)) = P(A) + P(B \setminus A).$$

Notice also that  $B = (B \setminus A) \cup (A \cap B)$ , where  $B \setminus A$  and  $A \cap B$  are disjoint sets. Then, using the third axiom, we can write the following:

$$P(B) = P((B \setminus A) \cup (A \cap B)) = P(B \setminus A) + P(A \cap B).$$

Further algebraic manipulation yields

$$P(B \setminus A) = P(B) - P(A \cap B).$$

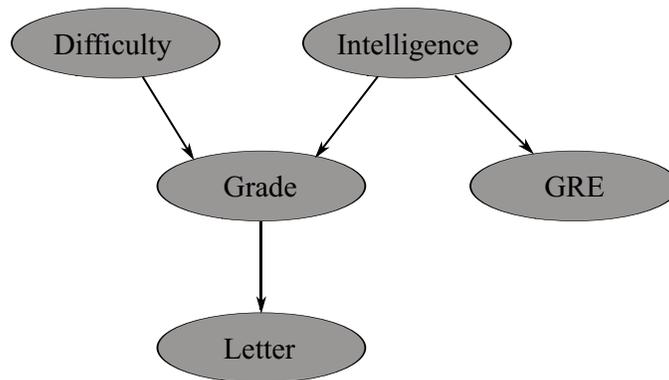
Substituting this to the first equation give the following result:

$$P(A \cup B)P(A) + P(B \setminus A) = P(A) + P(B) - P(A \cap B).$$

□

## Problem 3: Bayesian Networks

Consider the following Bayesian Network relating Sertac's performance in a class he is taking with Prof. Frazzoli, his GRE score, and the letter of recommendation he will be asking to apply for graduate school.



Write down the joint probability distribution of all the variables in this network in terms of the conditional probability distributions.

**Solution:** Let  $I$ ,  $D$ ,  $G$ ,  $T$ , and  $L$ , denote intelligence, difficulty, grade, test (GRE), and letter, respectively. Then,

$$\begin{aligned}
 P(I, D, G, T, L) &= P(L|I, D, G, T)P(I, D, G, T) \\
 &= P(L|G)P(I, D, G, T) \\
 &= P(L|G)P(T|I, D, G)P(I, D, G) \\
 &= P(L|G)P(T|I)P(I, D, G) \\
 &= P(L|G)P(T|I)P(G|I, D)P(I, D) \\
 &= P(L|G)P(T|I)P(G|I)P(I, D) \\
 &= P(L|G)P(T|I)P(G|I)P(I)P(D).
 \end{aligned}$$

## Problem 4: Bayesian Inference

You are leading a new airways company, but your company is in trouble. You have just been told that some of your aircraft have a problem with the electrical system. An estimate is that 0.5% of all your aircraft have this problem.

Luckily, you came across a pretty reliable (!) test that can detect this problem. You were told the following statistics. If an aircraft has the problem (i.e., the aircraft is broken), the test has correctly identified the problem in 99% of the cases. If, on the other hand, the aircraft does not have the problem (i.e., the aircraft is safe), then the test has correctly identified that there is no problem in 99% of the cases.

Given that the test identifies an aircraft of yours as broken, find the probability that the aircraft was actually safe. Also, given the test has identified an airplane of yours as safe, find the probability that the aircraft was actually broken. Is this test any useful? If yes, why? If not, what would improve it?

**Solution:** Let us denote the event that the aircraft is broken by  $B$ , and the event that the aircraft is safe by  $S$ . Also, denote the event that the test returns positive (i.e., the aircraft is broken) by  $+$  and the event that the test returns negative (i.e., the aircraft is safe) by  $-$ . Note the following.

- $P(B)$  is the probability that an aircraft is broken. Recall your statistics that 0.5% of your aircraft are broken. Thus, from your statistics you know that  $P(B) = 0.005$ .
- $P(S)$  is the probability that an aircraft is safe. This is basically  $P(S) = 1 - P(B) = 1 - 0.005 = 0.995$ .
- $P(+|B)$  probability that the test returns positive given that the aircraft is broken. Recall that you were told that this test correctly identifies the broken aircraft in 99% of the cases. Then,  $P(+|B) = 0.99$ .
- $P(-|B)$  probability that the test returns negative given that the aircraft is broken. This is basically  $P(-|B) = 1 - P(+|B) = 1 - 0.99 = 0.01$ .
- $P(-|S)$  probability that the test returns negative given that the aircraft is safe. Recall from the statistics that you were provided that the test correctly identifies the safe aircraft as safe in 99% of the cases. Then,  $P(-|S) = 0.99$ .
- $P(+|S)$  probability that the test returns positive given that the aircraft is safe. This is basically  $P(+|S) = 1 - P(-|S) = 1 - 0.99 = 0.01$ .

We are asked to compute the quantities  $P(S|+)$  and  $P(B|-)$ . Below, we do so using the Bayes' rule.

$$P(S|+) = \frac{P(+|S)P(S)}{P(+)} = \frac{P(+|S)P(S)}{P(+|S)P(S) + P(+|B)P(B)} = \frac{0.01 \times 0.995}{0.01 \times 0.995 + 0.99 \times 0.005} = 0.6678.$$

Hence, 66.78% of the aircraft that is identified as broken by test are actually safe and working perfectly fine! In other words, of those aircraft that is identified as broken by the test, only 33.22% of them are actually broken, the rest was working correctly. That is, if you were to discard all the aircraft that tested positive, then 66.78% of what you discard would be working aircraft. You can think of this as the *sacrifice* this testing is making, while trying to discard some of the broken aircraft.

You may think that despite the sacrifice, the test has solved your problems and now your aircraft are all safe. Let us compute the probability that an aircraft is actually broken, although it was declared safe by the test.

$$P(B|-) = \frac{P(-|B)P(B)}{P(B)} = \frac{P(-|B)P(B)}{P(+|S)P(S) + P(+|B)P(B)} = \frac{0.01 \times 0.005}{0.01 \times 0.995 + 0.99 \times 0.005} = 0.0033.$$

This shows that 0.33% of your aircraft still have the problem that you have started with. Recall that before using the test the percentage of broken aircraft 0.5%. Not much seems to have changed despite the big sacrifice after you use this seemingly accurate test.

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