Probes in the Orbital Motion-Limited (OML) Regime

Ref. Laframboise, J. G., and L. W. Parker. "Probe Design for Orbit-limited Current Collection." *Physics of Fluids* 16, no.5 (1973): 629-36.

For 2-D or 3-D objects collecting ions or electrons, the perturbation to the background plasma dies away over distance comparable to the object's size, and beyond that one can assume equilibrium conditions (this not so for plasma, 1-D, conditions). In equilibrium,

$$\text{at } \infty: \qquad f(\vec{w}) \;=\; n_\infty \left(\frac{m}{2\pi k T_\infty}\right)^{3/2} e^{-\frac{E_\infty}{kT_\infty}} \qquad \left(E_\infty \;=\; \frac{1}{2}m(w_x^2+w_y^2+w_z^2)\right) \tag{1}$$

If there are no collisions ($mfp >>$ object size), then Vlasov's equation applies, and it states that f does not change following a particle's trajectory. Hence f is a function of any of the constants of motion, and in particular, any function of the energy E satisfies Vlasov's equation. The particular function

$$f(\vec{w}) \;=\; n_\infty \left(\frac{m}{2\pi k T_\infty}\right)^{3/2} e^{-\frac{E}{kT_\infty}} \tag{2}$$

is one such, and so, if a trajectory connects to ∞, where (1) applies and serves as initial condition, this is the solution. Since $E = \frac{1}{2}mw^2 + q\phi$,

$$f(w) \;=\; n_\infty e^{-\frac{q\phi}{kT_\infty}} \left(\frac{m}{2\pi k T_\infty}\right)^{3/2} e^{-\frac{mw^2}{2kT_\infty}} \tag{3}$$

This is valid for any $\vec{w}$ which actually occurs, but there is no need for all $\vec{w}$'s to occur. This may happen for two reasons:

(a) For some $\vec{w}$'s (at a given $\vec{r}$) the energy may be negative, in which case the trajectory cannot come from ∞ (where $E = E_\infty = \frac{1}{2}mw^2 + q\phi$).
(b) Even if $E > 0$, some trajectories still do not connect to ∞, because they begin and end on the body surface. This is very dependent on geometry and field distribution.

By definition, the OML regime is that for which (b) does not happen on the body's surface, i.e., any energetically possible trajectory reaching the body originates at ∞. In that case, it is very easy to calculate the collected current for any shape of the collector. The only distinction to be made is whether the collector is 2-D or 3-D; this is because in 2-D the $\vec{w}$ component parallel to the axis ($w_\parallel$) is separately conserved, and so we need to impose $E_\perp > 0$ instead of $E > 0$ (if we imposed only $E > 0$, $E_\perp + E_\parallel > 0$, $E_\perp > -E_\parallel$, we would be admitting some trajectories with $0 > (w_\perp^2)_\infty > -(w_\parallel^2)_\infty$, which cannot be).

1. Repelling Bodies

Use (3) on the body surface, whenever

$$\begin{cases} \frac{1}{2}mw^2 + q\phi_p > 0 & (3-D) \\ \\ \frac{1}{2}mw_\perp^2 + q\phi_p > 0 & (2-D) \end{cases} \tag{4}$$

For the repulsive case q and ϕ_p have the same sign, and (4) or (5) is automatic, so there is no restriction. Hence, in this case, shape does not matter at all. We have an arriving flux

$$J = \iiint |w_n| f(w) d^3w \tag{5}$$

Using a local polar coordinate system, and counting all one-sided arriving velocities

$$J = n_\infty e^{-\frac{q\phi_p}{kT_\infty}} \left(\frac{m}{2\pi kT_\infty}\right)^{3/2} \underbrace{\int_0^\infty dw e^{-\frac{mw^2}{2kT_\infty}} w^3 \underbrace{\int_0^{\pi/2} cos\theta sin\theta 2\pi d\theta}_{2\pi\frac{1}{2}=\pi}}_{\pi\left(\frac{2kT_\infty}{m}\right)^2 \int_0^\infty x^{3/2}e^{-x}\frac{1}{2}x^{-1/2}dx=\pi\left(\frac{2kT_\infty}{m}\right)^2\frac{1}{2}\overbrace{\Gamma(2)}^{1}=\frac{1}{2\pi}\left(\frac{2\pi kT_\infty}{m}\right)^2}$$

$$\frac{mw^2}{2kT_\infty} = x \qquad w = \left(\frac{2kT_\infty}{m}\right)^{1/2} x^{1/2}$$

$$J = \frac{n_\infty}{2\pi} e^{-\frac{q\phi_p}{kT_\infty}} \left(\frac{2\pi kT_\infty}{m}\right)^{1/2} \qquad \boxed{J = \frac{n_\infty \bar{c}_\infty}{4} e^{-\frac{q\phi_p}{kT_\infty}}} \qquad \left(\bar{c}_\infty = \sqrt{\frac{8}{\pi}\frac{kT_\infty}{m}}\right) \tag{6}$$

Notice this gives the same current density at any point on the (equipotential) body, for any body shape (in the OML regime).

2. 3-D Attracting Body (electrons to $\phi_p > 0$, ions to $\phi_p < 0$)

$$\text{Impose} \qquad \frac{1}{2}mw^2 + q\phi_p > 0 \tag{7}$$

and since $\phi_p < 0$, now, $w > \sqrt{\frac{2|q\phi_p|}{m}}$ and repeat calculation

$$J = n_\infty e^{-\frac{q\phi_p}{kT_\infty}} \left(\frac{m}{2\pi kT_\infty}\right)^{3/2} \pi \underbrace{\int_{\sqrt{-\frac{2q\phi_p}{m}}}^\infty w^3 e^{-\frac{mw^2}{2kT_\infty}} dw}_{\frac{1}{2}\left(\frac{2kT_\infty}{m}\right)^2 \int_{\frac{-q\phi_p}{kT_\infty}>0}^\infty xe^{-x}dx=\frac{1}{2}\left(1+\frac{|q\phi_p|}{kT_\infty}\right)e^{\frac{q\phi_p}{kT_\infty}}}$$

$$x = \frac{mw^2}{2kT_\infty} > \frac{-q\phi_p}{kT_\infty}$$

$$\int_a^\infty xe^{-x}dx = -xe^{-x}|_a^\infty + \int_a^\infty e^{-x}dx = -e^{-x}(x+1)|_a^\infty = (a+1)e^{-a}$$

$$\boxed{J = \frac{n_\infty \bar{c}_\infty}{4}\left(1 + \frac{|q\phi_p|}{kT_\infty}\right)} \tag{8}$$

3. <u>2-D Attracting Body</u>

$$\text{Impose} \qquad \frac{1}{2}mw_\perp^2 + q\phi_p > 0 \qquad , \qquad w_\perp > \sqrt{\frac{2|q\phi_p|}{m}} \tag{9}$$

and now use a local cylindrical system at the surface:

$$J = \int_{w_\perp=\sqrt{\frac{2|q\phi_p|}{m}}}^\infty n_\infty(..)^{3/2} \int_{\theta=-\frac{\pi}{2}}^{\pi/2} e^{-\frac{mw_\perp^2 - 2|q\phi_p|}{2kT_{infty}}} (w_\perp cos\theta)(w_\perp dw_\perp d\theta) \underbrace{\int_{-\infty}^\infty e^{-\frac{mw_\parallel^2}{2kT_\infty}} dw_\parallel}_{\left(\frac{2\pi kT_\infty}{m}\right)^{1/2}}$$

$$J = 2n_\infty \left(\frac{2\pi kT_\infty}{m}\right)^{-1} \underbrace{\left(\int_{\sqrt{\frac{2|q\phi_p}{m}}}^\infty dw_\perp\right)}_{\left(\frac{2kT_\infty}{m}\right)^{3/2} \int_{\sqrt{\frac{|q\phi_p|}{kT_\infty}}}^\infty x^2e^{-x^2}dx}$$

$$\int_a^\infty x^2e^{-x^2}dx = -\frac{1}{2}xe^{-x^2}|_a^\infty + \frac{1}{2}\int_a^\infty e^{-x^2}dx = \frac{1}{2}ae^{-a^2} + \frac{1}{2}\frac{\sqrt{\pi}}{2}erfc(a)$$

$$J = n_\infty \frac{2}{2\pi}\sqrt{\frac{2kT_\infty}{m}} e^{\frac{|q\phi_p|}{kT_\infty}} \left[\left|\frac{q\phi_p}{kT_\infty}\right| e^{-\frac{|q\phi_p|}{kT_\infty}} + \frac{\sqrt{\pi}}{2} erfc\sqrt{\frac{|q\phi_p|}{kT_\infty}}\right]$$

$$\boxed{J = \frac{n_\infty \bar{c}_\infty}{4}\frac{2}{\sqrt{\pi}}\left[\frac{|q\phi_p|}{kT_\infty} + \frac{\sqrt{\pi}}{2}e^{\frac{|q\phi_p|}{kT_\infty}} erfc\sqrt{\frac{|q\phi_p|}{kT_\infty}}\right]} \tag{10}$$

For strong potentials ($|q\phi_p| >> kT_\infty$),

$$\frac{\sqrt{\pi}}{2}e^{x^2}erfc(x) = e^{x^2}\int_x^\infty e^{-t^2}dt = e^{x^2}\int_x^\infty -\frac{1}{2t}d(e^{-t^2}) = e^{x^2}\left[-\frac{e^{-t^2}}{2t}|_x^\infty + \int_x^\infty e^{-t^2}\left(-\frac{dt}{2t^2}\right)\right] \rightarrow \frac{1}{2x}$$

and so the second term ($\sim \frac{1}{2x}$) is negligible with regard to the first (x):

$$J \simeq \frac{n_\infty \bar{c}_\infty}{4}\sqrt{\frac{|q\phi_\infty|}{kT_\infty}}\frac{2}{\sqrt{\pi}} \;=\; \frac{n_\infty}{2\sqrt{\pi}}\sqrt{\frac{8}{\pi}\frac{\cancel{kT_\infty}}{m}}\sqrt{\frac{|q\phi_p|}{\cancel{kT_\infty}}}$$

$$\boxed{J \simeq \frac{n_\infty}{\pi}\sqrt{\frac{2|q\phi_p|}{m}}} \qquad (|q\phi_p| >> KT_\infty) \tag{11}$$

(again, anywhere on the 2-D attracting body)

<u>Examples</u>:

1. Spherical probe in OML

$$I \;=\; \begin{cases} 4\pi R^2 \dfrac{n_{e_x}}{4}\left[-\bar{c}_e\left(1+\dfrac{|e|\phi_p}{kT_e}\right)-\bar{c}_i e^{\frac{|e|\phi_e}{kT_i}}\right] & (\phi_p > 0) \\ 4\pi R^2 \dfrac{n_{e_x}}{4}\left[-\bar{c}_i\left(1+\dfrac{e\phi_p}{kT_i}\right)-\bar{c}_e e^{\frac{|e\phi_e|}{kT_e}}\right] & (\phi_p < 0) \end{cases}$$

2. Cylindrical probe in OML (with $|e\phi_p| >> kT$)

$$I \;=\; \begin{cases} \simeq 2RLn_{e_x}\left[\sqrt{\dfrac{2(e\phi_p)}{m_e}} - \sim 0\right] & (\phi_p > 0) \\ \simeq 2RLn_{e_x}\left[\sqrt{\dfrac{2(e\phi_p)}{m_i}} - \sim 0\right] & (\phi_p < 0) \end{cases}$$

16.55 Ionized Gases
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