Motion of Charges in Uniform $\vec{E}$ and $\vec{B}$ Fields

Assume an ionized gas is acted upon by a uniform (but possibly time-dependent) electric field $\vec{E}$, and a uniform, steady magnetic field $\vec{B}$. These fields are assumed to be externally supplied, and to be unaffected by the presence or the motion of the charges themselves. In reality, the charges may generate significant space charge, and modify the potential according to Poisson's equation ($\nabla^2\phi = -\rho/\epsilon_0$), or generate significant net current, and hence modify $\vec{B}$ through Ampere's equation ($\nabla \times \vec{B}/\mu_0 = \vec{j} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}$). If so, the analysis in this section would have to be viewed at most as part of a more complete formulation that would incorporate these modifications in a self-consistent fashion.

For a particle of mass m and charge q, moving at velocity $\vec{w}$, the equation of motion is,

$$m\frac{d\vec{w}}{dt} = q(\vec{E} + \vec{w} \times \vec{B}) \tag{1}$$

The projection on $\vec{B}(w_{||}, E_{||})$ is simply,

$$m\frac{dw_{||}}{dt} = qE_{||} \tag{2}$$

and this part of the motion is unaffected by $\vec{B}$. We will concentrate here on the projection perpendicular to $\vec{B}(\vec{w}_\perp, \vec{E}_\perp)$. We can take advantage of the constancy of $\vec{B}$ and the fact that $\vec{E}_\perp$ is uniform, and use complex algebra to streamline the analysis. Take axis (O_{xyz}), where $\vec{B}$ lies along O_z, and define,

$$w = w_x + iw_y \tag{3}$$

$$E = E_x + iE_y \tag{4}$$

So that $\vec{w} \times \vec{B} = (w_x + iw_y) \times B\hat{i}_z = (w_y - iw_x)B = -iwB$.

The perpendicular part of (1) is then,

$$m\frac{dw}{dt} = q(E - iwB)$$

or,

$$\frac{dw}{dt} + i\omega_c w = \frac{q}{m}E \tag{5}$$

where,

$$\omega_c = \frac{qB}{m} \tag{6}$$

is the cyclotron frequency, or gyro frequency, of the particle.

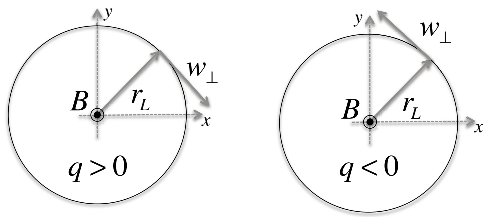
(a) Case with no electric field

The general solution of (5) when $E = 0$ is,

$$w = Ae^{-i\omega_c t} \tag{7}$$

and since $w = \frac{dz}{dt}$ with $z = x + iy$, then

$$z = i\frac{A}{\omega_c}e^{-i\omega_c t} \tag{8}$$



The constant A in (7) can be taken to be real (this defines $t=0$ as the time of crossing through the $y=0$ axis), and can be identified as the constant magnitude $A=w_\perp$ of the perpendicular velocity. From (8), the position vector of the particle is 90° rotated with respect to the velocity, and has a magnitude $r_L=\frac{A}{\omega_c}=\frac{w_\perp}{qB/m}$,

$$r_L \;=\; \frac{mw_\perp}{qB} \tag{9}$$

This is called the Larmor radius of the particle, for a velocity $w_\perp$. This is also called the gyro radius. Both vectors w and z are seen from (7) and (8) to rotate at angular velocity ω_c. The motion is in the sense shown in the figure above, depending on polarity.

The gyro frequency for electrons can be fairly high, usually second only to the plasma frequency, among the important scales. For example, in a Hall thruster, with $T_e=20eV=20\frac{e}{k}=20\times 11,600=232,000K$, the mean thermal speed is,

$$\bar{c}_e \;=\; \sqrt{\frac{8}{\pi}\frac{kT_e}{m_e}} \;=\; \sqrt{\frac{8}{\pi}\frac{1.38\times 10^{-23}\times 2.32\times 10^{5}}{9.11\times -31}} = 2.99\times 10^{6} ms^{-1}$$

With a magnetic field of the order of 0.01 Tesla, we find,

$$\omega_{ce} \;=\; \frac{eB}{m_e} = \frac{1.6\times 10^{-19}\times 0.01}{9.11\times 10^{-31}} = 1.8\times 10^{9} rad/s\ ;\ r_L \;=\; \frac{\bar{c}_e}{\omega_{ce}} = 1.7mm$$

The ions are much heavier, especially in propulsion devices using Xenon. From (6) and (9), $\omega_{ci}/\omega_{ce}=m_e/m_i$ and $r_{Li}/r_{Le}=\sqrt{m_i/m_e}$. In the Hall thruster example, with Xenon, $m_i/m_e=131\times 1850=2.42\times 10^5$, so $\omega_{ci}=7.4\times 10^3 rad/s=1.2kHz$ and $r_{Li}=0.84m$. Since the device is only a few cm in typical dimension, the ion trajectories are nearly straight in it (weak ion magnetization), but the electron trajectories (unless disrupted by collisions) are dominated by very tight gyro rotations (strong electron magnetization). These conclusions would be strongly modified if the ions were hydrogen and B were higher. For example, in fusion, $B\approx 5T$, and we find for H^+ ions at 1 keV ($\bar{c}_i=5\times 10^5 m/s$), $r_{Li}=1mm$, meaning that even the ions are now strongly magnetized.

From the sense of the gyro motion of both the electrons and ions, we see that this motion will induce in both cases some magnetic field in a direction opposite the original $\vec{B}$. This is called diamagnetism. For a general distribution of current densities $\vec{j}$, the induced $\vec{B}$ is,

$$\vec{B}_{ind} \;=\; \sum_{particles} (\mu_0\vec{\mu}) \tag{10}$$

where $\mu_0=4\pi\times 10^{-7} Hy/m$ is the permeability of vacuum and $\vec{\mu}$ is the magnetic moment, defined as,

$$\vec{\mu} \;=\; \frac{1}{2}\int\int\int \vec{r}\times\vec{j}dV \tag{11}$$

For an isolated gyrating charge, the current is concentrated along the circular path, and has a mean value $q/(period) = q\omega_c/2\pi$. Dividing by the infinitesimal area δA swept by q, we get j, the current density, and so,

$$\vec{r} \times \vec{j} \;=\; r_L \frac{q\omega_c}{2\pi} \frac{1}{\delta A} \left(\frac{-\vec{B}}{B} \right)$$

The volume to be integrated over is $2\pi r_L \delta A$, therefore,

$$\vec{\mu} \;=\; -\frac{1}{2} r_L^2 \frac{q\omega_c}{2\pi} \frac{1}{\delta A} 2\pi \delta A \left(\frac{\vec{B}}{B} \right) = -\frac{1}{2} \left(\frac{m w_\perp}{qB} \right)^2 q \left(\frac{qB}{m} \right) \left(\frac{\vec{B}}{B} \right)$$

or in magnitude,

$$\mu \;=\; \frac{\frac{1}{2} m w_\perp^2}{B} \tag{12}$$

where the numerator is the kinetic energy of the perpendicular motion of the particle.

Assume n particles per unit volume are magnetized (usually, in Electric Propulsion, this is n_e, the electron density, but in a fusion plasma, $n = n_e + n_i$, as we saw). Then, from (10),

$$\vec{B}_{ind} \;=\; -n\mu_0 \mu \left(\frac{\vec{B}}{B} \right)$$

and, from the definition of temperature, $\left\langle \frac{1}{2} m w_\perp^2 \right\rangle_{particles} \;=\; 2\frac{1}{2} kT_\perp$, with the factor of 2 accounting for the two dimensions involved in $\vec{w}_\perp$,

$$\left| \vec{B}_{ind} \right| \;=\; \mu_0 n \frac{kT_\perp}{B}$$

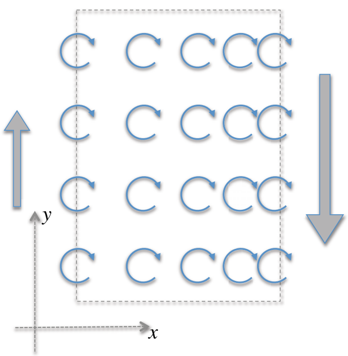
or,

$$\frac{\left| \vec{B}_{ind} \right|}{B} \;=\; \frac{nkT_\perp}{B^2/\mu_0} = \frac{(\text{kinetic pressure})_\perp}{\text{magnetic pressure}} \tag{13}$$

For the Hall thruster example, with $n = n_e \approx 10^{18} m^{-3}$, $T_\perp = 20eV$, $B = 0.01T$, $B_{ind}/B \approx 4 \times 10^{-3}$, which can be ignored. For fusion, $n \approx 10^{19} m^{-3}$, $T_\perp \approx 10^3 eV$, $B \approx 5T$, so $B_{ind}/B \approx 8 \times 10^{-5}$, also weak diamagnetism. Diamagnetic effects could be large in an electric propulsion plume propagating relatively far from the source, under the effect of the geomagnetic B-field ($\approx 3 \times 10^{-5} T$). Assuming $n \approx 10^{15} m^{-3}$, $T_\perp \approx 1eV$, we find $B_{ind}/B \approx 0.2$, so that the plume will tend to exclude the ambient field.

A non-uniform distribution of magnetic moments, such as from a density gradient, gives rise collectively to a macroscopic current, called magnetization current or diamagnetic current.

An illustration is shown in the figure below: the elementary gyro currents in the vertical direction cancel pairwise inside the box, but there is a net upwards current on the left edge, and a greater downwards current on the right edge. Overall, a region with a positive dn/dx, as indicated, sees a negative j_y due to this effect. A more detailed analysis follows.

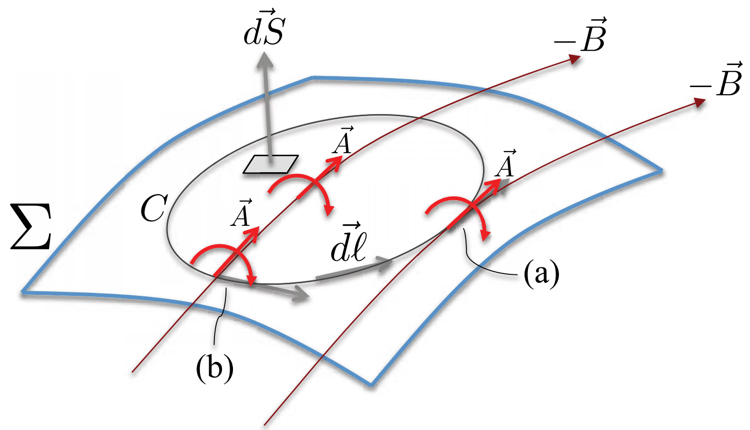


We begin by recasting the magnetic moment in the form,

$$\vec{\mu} \;=\; -\frac{1}{2} r_L^2 I 2\pi \left(\frac{\vec{B}}{B}\right) = -IA\frac{\vec{B}}{B} = +I\vec{A} \tag{14}$$

where $A = \pi r_L^2$ is the area of the small gyro loop, and, as before, $I = q/(\text{gyro period})$.

Consider a macroscopic surface Σ containing a contour C, and let us calculate the net current which crosses Σ due to the gyro motion of the particles lying near Σ. Only particles actually near the contour C itself contribute, since otherwise the current loops come into and out of the enclosed portion of Σ. When $\vec{A}$ is parallel to the line element $\vec{d\ell}$, the particle contributes a net passing current I (case (a) in the figure). When $\vec{A}$ is perpendicular to $\vec{d\ell}$ (case (b)) there is no contribution.



Overall, any particle in the volume element $\vec{A}\cdot\vec{d\ell} = A_{\perp C} d\ell$ contributes I to I_{crossing},

$$I_{\text{crossing}} \;=\; \oint nI\vec{A}\cdot\vec{d\ell} = \oint \vec{M}\cdot\vec{d\ell} \tag{15}$$

where $\vec{M} = n\vec{\mu} = nI\vec{A}$ is the so-called Magnetization vector. Converting to a surface integral,

$$I_{\text{crossing}} \;=\; \int\!\!\int_{\Sigma} \left(\nabla\times\vec{M}\right)\cdot\vec{dS} \tag{16}$$

where dS is a surface element on Σ, and the vector $\vec{dS}$ points perpendicular to Σ. Clearly, the magnetization current density is then,

$$\vec{j}_M \;=\; \nabla\times\vec{M} \tag{17}$$

This is in addition to other currents which may arise when the "guiding centers", i.e., the centers of the Larmor circuits, move about. These guiding-center motions are called drifts, and will be examined later. A physical interpretation of $\vec{M}$ is provided by rewriting the induced magnetic field as,

$$\vec{B}_{ind} \;=\; +\mu_0\vec{M} \tag{18}$$

so that $+\mu_0\vec{M}$ is the contribution to $\vec{B}$ due to elementary gyro motions.

Consider now a "quasi-static" situation, where frequencies are low enough that $\left|\epsilon_0\frac{\partial\vec{E}}{\partial t}\right| << \vec{j}$. This turns out to be applicable to all our problems of interest, except EM wave propagation. We then have,

$$\nabla\times\frac{\vec{B}}{\mu_0} = \vec{j} \tag{19}$$

where $\vec{j}$ is the total current density. If we separate this into the drift current $\vec{j}_d$ and the magnetization current,

$$\nabla\times\frac{\vec{B}}{\mu_0} = \vec{j}_d+\nabla\times\vec{M}$$

or

$$\nabla\times\left(\frac{\vec{B}}{\mu_0}-\vec{M}\right) = \vec{j}_d$$

This prompts the introduction of a vector $\vec{H}$ (the magnetic field strength) such that,

$$\vec{H} = \frac{\vec{B}}{\mu_0}-\vec{M} \tag{20}$$

and then,

$$\nabla\times\vec{H} = \vec{j}_d$$

Since in our situation $\vec{M}$ is itself proportional to $\vec{B}$,

$$\vec{M} = +\frac{\vec{B}_{ind}}{\mu_0} = -n\mu\frac{\vec{B}}{B} = -n\frac{\left\langle\frac{1}{2}mw_\perp^2\right\rangle}{B^2}\vec{B}$$

we can define a modified permeability of the medium, μ_m, such that (20) can be written,

$$\vec{H} = \frac{\vec{B}}{\mu_m} \tag{21}$$

where,

$$\frac{1}{\mu_m} = \frac{1}{\mu_0}+\frac{n\left\langle\frac{1}{2}mw_\perp^2\right\rangle}{B^2} = \frac{1}{\mu_0}+\frac{P_\perp}{B^2} \tag{22}$$

so, in this (diamagnetic) case, $\mu_m << \mu_0$. The opposite is true of ferromagnetic media, where $\vec{B}_{ind}$ is in the same direction as $\vec{B}$. More directly, we can still use μ_0, but remember to count both drift and magnetization currents.

(b) Effect of a uniform electric field (steady)

Returning to Eq. (1) $d\vec{w}/dt = \frac{q}{m}\left(\vec{E}+\vec{w}\times\vec{B}\right)$, we notice that for that velocity $\vec{w}=\vec{v}_D$ such that $\vec{E}+\vec{v}_D\times\vec{B}=0$, the electrostatic and the magnetic force cancel each other, and $d\vec{v}_D/dt=0$, which is consistent with that cancellation. To solve for $\vec{v}_D$, we write,

$$\left(\vec{E}+\vec{v}_D\times\vec{B}\right)\times\vec{B} = 0$$

$$\vec{E} \times \vec{B} + \vec{B}\left(\vec{v}_D \cdot \vec{B}\right) - \vec{v}_D\left(\vec{B} \cdot \vec{B}\right) \;=\; 0$$

The part of $\vec{v}_D$ parallel to $\vec{B}$ is non-zero only if $\vec{E}_{||} = 0$ (otherwise the particles accelerate along $\vec{B}$ so $\vec{v}_D$ is no longer constant). So, assume $\vec{v}_D$ is $\perp$ to $\vec{B}$, and $\vec{v}_D \cdot \vec{B} = 0$, with the result that,

$$\vec{v}_D \;=\; \frac{\vec{E} \times \vec{B}}{B^2} \tag{23}$$

which is a constant drift velocity perpendicular to both $\vec{E}$ and $\vec{B}$, and independent of the charge, the mass or the charge's sign. In magnitude,

$$|\vec{v}_D| \;=\; \frac{E}{B} \tag{24}$$

This result can also be obtained from the complex formulation of Eq. (5). With $E =$ constant, the equation admits a particular solution,

$$i\omega_c w \;=\; \frac{q}{m}E$$

$$w \;=\; v_D = -i\frac{q}{m\omega_c}E$$

and since $\omega = qB/m$ then,

$$v_D \;=\; -i\frac{E}{B} \tag{25}$$

which is equivalent to (23). Adding to this an arbitrary amount of the homogeneous solution $w_H = e^{-i\omega_c t}$, the general solution is,

$$w \;=\; Ae^{-i\omega_c t} - i\frac{E}{B} \tag{26}$$

which is a superposition of the $\vec{E} \times \vec{B}$ drift and Larmor rotations.

The nature of this total motion is best understood for a simple case when $E = iE_y$, giving $w = Ae^{-i\omega_c t} + E_y/B$, with a real, positive drift along x (real E_y/B). Suppose we impose $w = 0$ at $t = 0$, then $A = -E_y/B$ and $w = \frac{E_y}{B}\left(1 - e^{-i\omega_c t}\right)$.

Real part:

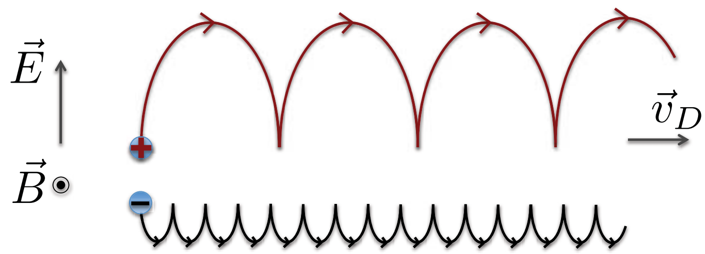
$$w_x \;=\; \frac{E_y}{B}\left[1 - \cos(\omega_c t)\right] \;\rightarrow\; x \;=\; \frac{E_y}{\omega_c B}\left[\omega_c t - \sin(\omega_c t)\right]$$

Imaginary part:

$$w_y \;=\; \frac{E_y}{B}\sin(\omega_c t) \;\rightarrow\; y \;=\; \frac{E_y}{\omega_c B}\left[1 - \cos(\omega_c t)\right]$$

$$y_{max} \;=\; \frac{2E_y}{\omega_c B} \;=\; \frac{2mE_y}{qB^2}$$

So, w_x is always positive, except it becomes zero at $\omega_c t = 0, 2\pi, 4\pi, ...$ and w_y is zero at $\omega_c t = 0, \pi, 2\pi, 3\pi, 4\pi, ...$ and oscillates (+) and (−). For $q > 0$ (heavy ion) and $q < 0$ (light electron), the motion is sketched below:



The figure shows why the drift is in the same direction regardless of the charge's sign. Ions and electrons start out under $\vec{E}$ alone (zero velocity gives zero magnetic force) in opposite directions. Then, as the vertical velocity develops (with opposite signs), the magnetic force $q\vec{w} \times \vec{B}$ turns the trajectories to the right in both cases, since $qv_y > 0$ for both. We can also see that the penetration in the y direction is much greater for the ions, and that the E_y field does separate the charges, as one would expect, but only to a finite extent (with no $\vec{B}$ field, they would simply accelerate away from each other forever).

In a Hall thruster, where $r_{Li} >> r_{Le}$ and $r_{Li} >> L$, only the electrons drift. In a fusion plasma, both ions and electrons do, and as a consequence the $\vec{E} \times \vec{B}$ drift current is zero. In the Hall thruster case, the ion current in the $\vec{E} \times \vec{B}$ direction is nearly zero, but electrons drift at the full E/B, and there is a nonzero $\vec{E} \times \vec{B}$ drift current, which is called the Hall current:

$$\vec{j}_{Hall} = -en_e \frac{\vec{E} \times \vec{B}}{B^2} \tag{27}$$

(c) Effect of a time-varying (but uniform) $\vec{E}$ field

Returning to Eq. (5), if $E = E(t)$, the general solution can be obtained by the method of variation of the "constant" in the homogenous part. The result is,

$$w = \left[A + \int^t \frac{q}{m} E e^{i\omega_c t'} dt' \right] e^{-i\omega_c t} \tag{28}$$

The integral can be evaluated explicitly for sinusoidal $E(t)$ of any frequency. But before looking at that case, consider the case where $E(t)$ has any shape, but has a slow variation, when compared to the cyclotron motion. Since ω_c is very high for electrons in general, these "slow" changes may not be very slow after all. A formal solution can then be obtained using integration by parts in (28). Using,

$$e^{i\omega_c t} dt = \frac{m}{iqB} d\left(e^{i\omega_c t}\right) \text{ then,}$$

$$w = \left[A + \frac{E}{iB} e^{i\omega_c t} - \frac{1}{iB} \int^t e^{i\omega_c t'} \frac{dE}{dt'} dt' \right] e^{-i\omega_c t}$$

and repeating the process,

$$w = Ae^{-i\omega_c t} - i\frac{E}{B} + \frac{i}{B}\frac{m}{iqB}\frac{dE}{dt} + \ldots$$

$$w = Ae^{-i\omega_c t} - i\frac{E}{B} + \frac{m}{qB^2}\frac{dE}{dt} + ... \tag{29}$$

Higher terms would contain higher derivatives of E, and are neglected for these "slow" variations. The first two terms are familiar (gyro motion, $\vec{E}\times\vec{B}$ drift). The third term is a new drift, called Polarization drift,

$$\vec{v}_p = \frac{m}{qB^2}\frac{d\vec{E}}{dt} \tag{30}$$

The reason of this name is the following: As seen from (30), ions and electrons will drift in opposite directions, and so there will always be a current associated with this drift. Assuming $n_e = n_i$, $q_e = -e$, and $q_i = e$, this current is,

$$\vec{j}_p = e\vec{v}_{pi}n_e - e\vec{v}_{pe}n_e = en_e\left(\frac{m_i}{eB^2} + \frac{m_e}{eB^2}\right)\frac{d\vec{E}}{dt}$$

The m_e part is negligible, so the ion polarization current dominates. The overall plasma mass density is $\rho_p = n_e(m_i + m_e) \approx n_e m_i$, so,

$$\vec{j}_p = \frac{\rho_p}{B^2}\frac{d\vec{E}}{dt} \tag{31}$$

This is the same form as Maxwell's polarization (or displacement) current $\vec{j}_{Maxw} = \epsilon_0\partial\vec{E}/\partial t$, hence the name. It is natural to ask whether $\vec{j}_p$ is or is not much larger than $\vec{j}_{Maxw}$, since $\vec{j}_{Maxw}$ is usually negligible. Their ratio is $\rho_p/\epsilon_0 B^2$; for a Hall thruster plasma ($n_e \approx 10^{18}m^{-3}$, $m_i = m_{Xe} = 2.2\times10^{-25}kg$, $B\approx 0.01T$) we find,

$$\left|\frac{\vec{j}_p}{\vec{j}_{Maxw}}\right| = \frac{\rho_p}{\epsilon_0 B^2} \approx \frac{10^{18}\times 2.2\times10^{-25}}{8.854\times10^{-12}\times(10^{-2})^2} = 2.5\times10^8$$

This is a clear warning that the displacement current (31) may not be negligible, even when Maxwell's is. Carrying the thought forward, since we know that, in vacuum, Maxwell's currents are responsible for EM waves, we now realize that in a plasma, where $\vec{j}_p$ replaces $\vec{j}_{Maxw}$, there should be formally identical waves due to $\vec{j}_p$. To see this, assume $\vec{j}_p$ is the only current present, and ignore $\vec{j}_{Maxw}$:

$$\nabla\times\frac{\vec{B}}{\mu_0} \simeq \frac{\rho_p}{B^2}\frac{\partial\vec{E}}{\partial t} \tag{32}$$

For small perturbations, $\vec{B} = \vec{B}_0 + \vec{B}'$ and $\rho_p = \rho_{p0} + \rho_p'$. Since $B' << B_0$ and assuming there is no electric field in the background $\vec{E} = \cancel{\vec{E}_0} + \vec{E}'$, Eq. (32) reads,

$$\nabla\times\frac{\vec{B}'}{\mu_0} = \frac{\rho_{p0}}{B_0^2}\frac{\partial\vec{E}'}{\partial t}$$

Take the curl of this equation, and use Faraday's law $\nabla\times\vec{E}' = -\partial\vec{B}'/\partial t$,

$$\nabla\times\left(\nabla\times\frac{\vec{B}'}{\mu_0}\right) = -\frac{\rho_{p0}}{B_0^2}\frac{\partial^2\vec{B}'}{\partial t^2}$$

and since $\nabla \times (\nabla \times \vec{B}') = \nabla\cancel{(\nabla \cdot \vec{B}')} - \nabla^2 \vec{B}'$, we obtain the wave equation,

$$\frac{\partial^2 \vec{B}'}{\partial t^2} = \frac{B_0^2}{\mu_0 \rho_{p0}} \nabla^2 \vec{B}' \tag{33}$$

with a propagation velocity,

$$v_A = \sqrt{\frac{B_0^2}{\mu_0 \rho_{p0}}} \tag{34}$$

which is called the <u>Alfven velocity</u>. These waves are, like ordinary EM waves, <u>transverse</u>, in the sense that $\vec{E}'$ and $\vec{B}'$ are perpendicular to the propagation vector, and to each other. Since we found that $\rho_{p0}/\epsilon_0 B_0^2 >> 1$, we can put,

$$\frac{\mu_0 \rho_{p0}/B_0^2}{\epsilon_0 \mu_0} >> 1 \quad \text{or} \quad \frac{c^2}{v_A^2} >> 1$$

i.e., the Alfven velocity is generally well below the speed of light.

Note: If we include the charge motion along $\vec{B}_0$, we can obtain a different type of "compressive" Alfven waves.

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