

Radiation Transport in a Gas

By analogy to a particle gas, define a photon distribution function by,

$$f_\nu(\nu, \vec{\Omega}; \vec{r}, t) d\nu d\Omega d^3r = \text{Number of photons of a frequency in } (\nu, \nu + d\nu),$$

in a volume at $\vec{r}(d^3r)$, with direction in $(\vec{\Omega}, d\Omega)$ (1)

A related quantity, which is actually the one used most often, is the Spectral Radiation Intensity, defined as,

$$I_\nu = h\nu f_\nu c \quad (W/m^2/\text{sterad}/\text{Hz}) \quad \text{or} \quad (J/m^2/\text{sterad}) \quad (2)$$

so $I_\nu =$ Energy crossing unit area per unit time, normal to $\vec{\Omega}$ within a solid angle. Another related quantity is the Energy Density:

$$u_\nu = \int f_\nu h\nu d\Omega \quad (J/m^3/\text{Hz}) \quad (3)$$

which can also be written as,

$$u_\nu = \int \frac{I_\nu}{c} d\Omega \quad (4)$$

Finally, the Energy Flux Vector (or radiant heat flux) is defined as,

$$\vec{S}_\nu = \int f_\nu h\nu c \vec{\Omega} d\Omega = \int I_\nu \vec{\Omega} d\Omega \quad (5)$$

with units of $W/m^2/\text{Hz}$. For isotropic radiation, $\vec{S}_\nu = 0$, even though u_ν may not be zero (or I_ν). For the definition (5), we count the individual energy fluxes of all photons, of any direction, then we add them vectorially.

Equilibrium Distribution

This was derived from Bose-Einstein statistics, and it constitutes an example of an isotropic radiation field that can be obtained, say, inside a blacked enclosure at constant temperature. The various Equilibrium (or Black Body) quantities are

$$f_\nu = \frac{2\nu^2}{c^3} \frac{1}{e^{\frac{h\nu}{kT}} - 1} \quad (6)$$

$$u_\nu = 4\pi h\nu f_\nu = 8\pi \frac{h\nu^3}{c^3} \frac{1}{e^{\frac{h\nu}{kT}} - 1} \quad (7)$$

$$I_\nu = cf_\nu h\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT}} - 1} \equiv B_\nu \quad (8)$$

Notice u_ν or I_ν are maximum when,

$$\lambda T = 2900(\mu m) \times K \quad (\text{Wein's Law}) \quad (9)$$

The total energy density in the radiation field is,

$$u = \int_0^\infty u_\nu d\nu = \frac{8\pi^5 k^4 T^4}{15 (hc)^3} \quad (10)$$

and the one-sided total energy flux is,

$$q = \frac{uc}{4} = \frac{2\pi^5 k^4 T^4}{15 h^3 c^2} = \sigma T^4 \quad (\sigma = 5.67 \times 10^{-8} \text{ W/m}^2/\text{K}^4) \quad (11)$$

But, of course, the net (two-sided) energy flux is zero. The relevance of q is that it is the power flux emitted from the surface of a black body, where only one side exists.

The Radiation Transport Equation

This is the analogue of the Boltzmann equation for photons. There are no forces on a photon, so all we have is,

$$\frac{\partial I_\nu}{\partial t} + \vec{c} \cdot \nabla I_\nu = \left(\frac{\partial I_\nu}{\partial t} \right)_{\text{collisions}} \quad (12)$$

(notice the equation could be in terms of f_{nu} , but I_ν is customary). Also, $\vec{c} = c\vec{\Omega}$. Comparing the terms on the left, $\frac{(\partial I_\nu / \partial t)}{(\vec{c} \cdot \nabla I_\nu)} \sim \frac{\Delta x}{c \Delta t} \ll 1$ usually. So, ignore “photon accumulation”:

$$c\vec{\Omega} \cdot \nabla I_\nu = \left(\frac{\partial I_\nu}{\partial t} \right)_{\text{collisions}} \quad (13)$$

or, in terms of distance s along the beam ($ds = cdt$),

$$\boxed{\vec{\Omega} \cdot \nabla I_\nu = \left(\frac{\partial I_\nu}{\partial t} \right)_{\text{collisions}}} \quad (14)$$

Structure of the Collision Operator

Photons can be absorbed, scattered, or emitted along their paths. Absorption + Scattering is also called Extinction. We ignore for now scattering.

Define the Spectral Absorption Coefficient k_ν , such that $k_\nu I_\nu =$ Energy absorbed in the unit spectral range about ν , per unit length along the $\vec{\Omega}$ -directed path, per steradian k_ν has dimensions of inverse length. In fact, $\frac{1}{k_\nu} = \lambda_\nu$ is the Photon Mean Free Path. Including scattering $k_\nu \rightarrow$ Extinction coefficient. Define also the Spectral Emission per unit length and per unit frequency, into unit solid angle about $\vec{\Omega}$ (J_ν).

With these definitions, (14) becomes

$$\boxed{\vec{\Omega} \cdot \nabla I_\nu = J_\nu - k_\nu I_\nu} \quad (15)$$

We must relate k_ν and J_ν to atomic properties of the gas through which the photons propagate.

Let N_n = Number of atoms in n^{th} excitation level.

Let α_{nm} = Cross section for a photon-induced $n \rightarrow m$ transition.

where the frequency ν is such that $h\nu = \epsilon_m - \epsilon_n$. Because of uncertainty and various Broadening mechanisms (see later), the energy level difference $\epsilon_m - \epsilon_n$ is spread over some finite (although often narrow) part of the spectrum, rather than it being sharply defined, and so several transitions may contribute to a particular frequency ν :

$$k_\nu = \sum_{n < m, \text{all } m} k_{nm} = \sum_{n < m, m} N_n \alpha_{nm} \quad (\text{with } \nu = (\epsilon_m - \epsilon_n)/h) \quad (16)$$

The emission can be spontaneous (independent of the radiation field), or it can be induced or stimulated (proportional to J_ν , and in the same direction). We introduce a spontaneous emission coefficient, β_{mn} , and an induced emission coefficient, γ_{mn} , and write,

$$J_\nu - k_\nu I_\nu = \sum_{n < m, m} (-N_n \alpha_{nm} I_\nu + \beta_{mn} N_m + \gamma_{mn} N_m I_\nu) \quad (17)$$

This would add up to zero for any (m, n) if the radiation were Black Body radiation (equilibrium):

$$(I_\nu)_{\text{equil.}} = B_\nu = \frac{B_{mn} N_m}{\alpha_{nm} N_n - \gamma_{mn} N_m} = \frac{\frac{\beta_{mn}}{\gamma_{mn}}}{\frac{\alpha_{nm} N_n}{\gamma_{mn} N_m} - 1} \quad (18)$$

Comparing this to $B_\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}$, we conclude that,

$$\boxed{\frac{\beta_{mn}}{\gamma_{mn}} = \frac{2h\nu^3}{c^2}} \quad ; \quad \boxed{\frac{\alpha_{nm}}{\gamma_{mn}} \left(\frac{N_n}{N_m} \right)_{\text{equil.}}} \quad (19a; 19b)$$

Now, in equilibrium we already know that,

$$\left(\frac{N_n}{N_m} \right)_{\text{equil.}} = \frac{g_n}{g_m} e^{+\frac{h\nu}{kT}} \quad (20)$$

and so (19b) simplifies to,

$$\boxed{\frac{\alpha_{nm}}{\gamma_{mn}} = \frac{g_m}{g_n}} \quad (21)$$

Equations (19a) and (21) are Einstein's relationships, and, as usual, allow us to express all three kinetic coefficients (α, β, γ) in terms of only one of them. This is sometimes chosen to

be absorption (α), other times spontaneous emission (β). Substituting Einstein's relations into (17), and choosing to express everything in terms of absorption, we obtain,

$$\boxed{\vec{\Omega} \cdot \nabla I_\nu = \sum_{n < m, m} \alpha_{nm} N_n \left[\frac{N_m g_n}{N_n g_m} \left(\frac{2h\nu^3}{c^2} + I_\nu \right) - I_\nu \right]} \quad (22)$$

Particular Cases

(a) For low enough frequencies ($\nu \ll \left(\frac{c^2 I_\nu}{2h}\right)^{1/3}$), or, at a given frequency, for strong enough radiation, we can neglect the ν^3 term (arising from spontaneous emission), and retain only stimulated emission:

$$\vec{\Omega} \cdot \nabla I_\nu = \sum_{n < m, m} N_n \alpha_{nm} \left(\frac{N_m/g_m}{N_n/g_n} - 1 \right) I_\nu \quad (23)$$

and if upper levels are more populated (in proportion to their degeneracy) than lower levels, the intensity increases along the beam path. This is amplification, or LASER action. Notice that UV lasers are more difficult than MW (MASER) or visible laser (higher ν requires higher I_ν), and X-ray lasers are only possible in the enormous radiation field of a nuclear fireball. Notice also that $\frac{N_m/g_m}{N_n/g_n}$ would be $e^{-h\nu/kT}$, 1 if the levels were equilibrated at some "population temperature" T . An inversion is necessary that will over-populate the upper levels; this is sometimes described as a "negative temperature".

(b) Gas energy levels populated according to some T , any frequency. This implies a partial equilibrium situation, in which the atomic/molecular levels are equilibrated among themselves, but the radiation field is out of equilibrium. This situation is very common, and is called Local Thermodynamic Equilibrium (LTE). Of course, no lasing is possible at LTE. In this case, as noted,

$$\begin{aligned} \frac{N_m g_n}{N_n g_m} &= e^{-\frac{h\nu}{kT}}, \text{ and so} \\ \vec{\Omega} \cdot \nabla I_\nu &= \underbrace{\left(\sum_{n < m, m} N_n \alpha_{nm} \right)}_{k_\nu} \underbrace{\left[\left(\frac{2h\nu^3}{c^2} + I_\nu \right) e^{-\frac{h\nu}{kT}} - I_\nu \right]}_{\frac{2h\nu^3}{c^2} e^{-\frac{h\nu}{kT}} - I_\nu (1 - e^{-\frac{h\nu}{kT}})} \\ \vec{\Omega} \cdot \nabla I_\nu &= k_\nu (1 - e^{-\frac{h\nu}{kT}}) \left[\frac{2h\nu^3/c^2}{e^{+\frac{h\nu}{kT}} - 1} - I_\nu \right] \end{aligned}$$

The first term in brackets is B_ν , the Black Body radiation intensity. Also, define a modified absorption coefficient (absorption minus stimulated emission) by,

$$k'_\nu = k_\nu (1 - e^{-\frac{h\nu}{kT}}) \quad (24)$$

and we obtain the LTE Radiation Transport Equation:

$$\boxed{\vec{\Omega} \cdot \nabla I_\nu = k'_\nu (B_\nu - I_\nu)} \quad (25)$$

Integrated relationships for a spectral line

For a line the absorption of the emission are concentrated, and the total spontaneous line emission is an atomic property. The decay rate from m to n is A_{mn} ; it can be distributed differently in ν , depending on T , collisions, etc., but the total is constant. Emission is isotropic, so per atom, $\frac{A_{mn}}{4\pi}$ per sterad. The net energy emission rate per unit volume, per sterad, for the whole line, is then $N_m \frac{A_{mn}}{4\pi} h\nu$. Comparing to the expression $(\Omega \cdot I_\nu)_{sp.em.} = \sum_{m,n < m} \beta_{mn} N_m$, which is per unit frequency, we see that,

$$\boxed{\int_{line} \beta_{mn} d\nu = \frac{A_{mn}}{4\pi} h\nu} \quad (26)$$

and then, from the Einstein relation between β_{mn} and α_{nm} , the integral over the line of the absorption cross-section is,

$$\boxed{\int_{line} \alpha_{nm} d\nu = \frac{c^2}{2h\nu^3} \frac{g_m}{g_n} \int_{line} \beta_{mn} d\nu = \frac{A_{mn}}{8\pi} \frac{g_m}{g_n} \frac{c^2}{\nu^2}} \quad (27)$$

and also, the integrated cross-section for stimulated emission is,

$$\boxed{\int_{line} \gamma_{mn} d\nu = \frac{g_m}{g_n} \int_{line} \alpha_{nm} d\nu = \frac{A_{mn}}{8\pi} \frac{c^2}{\nu^2}} \quad (28)$$

Solution along a ray

Say x is the distance along which we want to calculate the evolution of I_ν , and we know the state of the gas along $x(k'_\nu(x), B_\nu(x))$. The equation of propagation is,

$$\frac{dI_\nu}{dx} = k'_\nu(B_\nu - I_\nu) \quad (29)$$

and using variation of parameters, the solution is,

$$I_\nu(x) = e^{-\int_0^x k'_\nu(\varepsilon) d\varepsilon} \left[I_\nu(0) + \int_0^x k'_\nu(\varepsilon) B_\nu(\varepsilon) e^{\int_0^\varepsilon k'_\nu(\varepsilon_1) d\varepsilon_1} d\varepsilon \right] \quad (30)$$

The first factor is an attenuation factor. Inside the bracket, $I_\nu(0)$ is the intensity of the ray incident on the medium. The integral inside the brackets is the cumulative new radiation that has accumulated along the ray's path in $(0 - x)$, at the rate $k'_\nu B_\nu$ per unit length, except that the portions emitted "upstream" are attenuated only from their emission (ε) to the observation point (x). If the gas is uniform (constant k'_ν, B_ν), equation (30) reduces to,

$$I_\nu(x) = B_\nu + [I_\nu(0) - B_\nu] e^{-k'_\nu x} \quad (31)$$

Again, $I_\nu(0)e^{-k'_\nu x}$ is the remaining part of the original radiation, and $B_\nu(1 - e^{-k'_\nu x})$ is the accumulated gas emission.

Radiation “Signature” of a gas

If $I_\nu(0) = 0$, the intensity at $x = L$ is purely emitted radiation, given by

$$I_\nu(L) = B_\nu(1 - e^{-k'_\nu L}) \quad (32)$$

The quantity $k'_\nu L$ is called the “optical thickness” of the gas. Notice that this includes not only the actual thickness L , but also the absorption coefficient k'_ν , which can depend very strongly on frequency ν . Specifically, over some parts of the spectrum $k'_\nu L \ll 1$, and the gas is said to be optically thin there. Under these conditions, (32) can be approximated ($e^{-\epsilon} \simeq 1 - \epsilon$) as,

$$I_\nu(L) \simeq B_\nu k'_\nu L \quad (33)$$

This means we “see” a radiative spectrum that reflects proportionately the gas’ absorptivity k'_ν (which can now be interpreted as “emissivity”), with a slowly changing “black body envelope”, B_ν :

On the other hand, in the vicinity of a strong absorption line, we may have, for the same L , $k'_\nu L \gg 1$ and the gas is “optically thick” there. In that case, Equation (32) gives,

$$I_\nu(L) \simeq B_\nu \quad (34)$$

and we see, over a limited part of the spectrum, a black body radiator. Any structure k'_ν may have in this spectral region is lost due to the black-body saturation.

Application-Line Reversal Pyrometer

The temperature of a flame (or a moderate temperature plasma) can be measured to within $\pm 20K$ or so with the following arrangement: A calibrated tungsten ribbon, treated to emit as a black body, is placed behind the flame, and heated by an electrical current. The calibration must indicate the ribbon’s temperature, T_b , for a chosen current I . The ribbon is observed through the flame, using a dispersive element (spectroscope), to generate an observable spectrum. In this situation, Equation (31) applies, with $I_\nu(0) = B_\nu(T_b)$. We observe

$$I_\nu(L) = B_\nu(T) + [B_\nu(T_b) - B_\nu(T)]e^{-k'_\nu L} \quad (35)$$

As we vary T_b (with I), when $T_b = T$ we observe $I_\nu(L) = B_\nu(T_b)$, i.e., literally as if the flame were not there. When $T_b < T$, flame emission dominates, and in regions of the spectrum where $k'_\nu L \gg 1$ (thick), we see $I_\nu(L) \simeq B_\nu(T)$. It is customary to see the flame with some common salt, so that the yellow sodium doublet is definitively thick. As a consequence, when $T_b < T$, the strong yellow line is seen clearly against a darker background $B_\nu(T_b) + B_\nu(T)k'_\nu L$ in the “thin” regions.

Conversely, when $T_b > T$, the “yellow” line is actually dark ($B_\nu(T)$) against the thing background ($\sim B_\nu(T_b)$). The point of “inversion” is pretty sharply defined, and $T_b = T$ is easily found.

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