16.61

LECTURE # 5

- MOMENTUM, ANGULAR MOMENTUM
- DYNAMICS OF A SYSTEM OF PARTICLES.

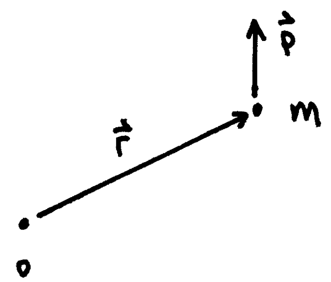
FURTHER BASICS

- LINEAR MOMENTUM $\qquad \vec{p} \triangleq m\dot{\vec{r}}^{I}$

- NEWTON'S LAW $\qquad \vec{F} = \dot{\vec{p}}^{I}$

  $\therefore$ IF $\vec{F} = 0$, $\vec{p}$ CONSTANT

- NOW TAKE THE MOMENT OF MOMENTUM (ANGULAR MOMENTUM)
  - MUST EXPLICITLY DEFINE A POINT ABOUT WHICH WE TAKE THE MOMENT.



- LET $\vec{H} \triangleq \vec{r} \times \vec{p}$

$|\vec{H}| \sim |p|$ PERPENDICULAR TO $\vec{r}$ TIMES MOMENT ARM $|\vec{r}|$

- DEFINE THE MOMENT OR TORQUE ABOUT o WITH FORCES $\vec{F}$ APPLIED TO m (CONSTANT)

$$\vec{M} = \vec{r} \times \vec{F} = \vec{r} \times (\dot{\vec{p}}^{I}) = m\,\vec{r} \times \ddot{\vec{r}}^{I}$$

BUT KNOW THAT $\dfrac{d^{I}}{dt}(\vec{r} \times \dot{\vec{r}}^{I}) = \cancel{\dot{\vec{r}}^{I} \times \dot{\vec{r}}^{I}} + \vec{r} \times \ddot{\vec{r}}^{I}$ $\quad$ 0 $\quad$ WHY?

$$\therefore \vec{M} = m\,\vec{r} \times \ddot{\vec{r}}^{I} = m\,\frac{d^{I}}{dt}(\vec{r} \times \dot{\vec{r}}^{I})$$

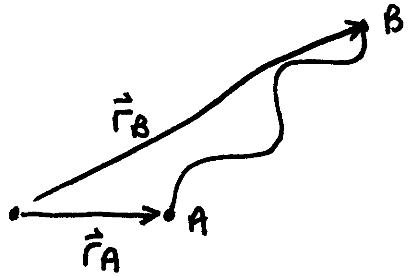
$$= \frac{d^{I}}{dt}\left(\vec{r} \times m\dot{\vec{r}}^{I}\right) = \frac{d^{I}}{dt}(\vec{r} \times \vec{p}) = \dot{\vec{H}}^{I}$$

- SO, IF $\vec{M} = 0$, THE $\dot{\vec{H}}^I = 0 \Rightarrow \vec{H} = \text{CONSTANT}$.

  $\therefore$ ANGULAR MOMENTUM UNCHANGED WHEN $M = 0$.

  – APPLIED MOMENT $\equiv$ TIME RATE OF CHANGE OF $\vec{H}$

- **WORK DONE** BY A FORCE ON A PARTICLE ?



$$W = \int_A^B \vec{F} \cdot d\vec{r} = m \int_A^B \ddot{\vec{r}}^I \cdot d\vec{r}$$

FORCE COMPONENT IN DIRECTION OF MOTION

– NOTES: (i) $d\vec{r} = \dot{\vec{r}}^I dt \quad \therefore \quad \ddot{\vec{r}}^I \cdot d\vec{r} = \ddot{\vec{r}}^I \cdot \dot{\vec{r}}^I dt$

(ii) CAN SIMPLIFY $\ddot{\vec{r}}^I \cdot \dot{\vec{r}}^I$ SINCE

$$\frac{d^I}{dt}\left(\dot{\vec{r}}^I \cdot \dot{\vec{r}}^I\right) = \ddot{\vec{r}}^I \cdot \dot{\vec{r}}^I + \dot{\vec{r}}^I \cdot \ddot{\vec{r}}^I = 2\, \ddot{\vec{r}}^I \cdot \dot{\vec{r}}^I$$

- SO

$$W = \frac{m}{2} \int_A^B \frac{d^I}{dt}\left(\dot{\vec{r}}^I \cdot \dot{\vec{r}}^I\right) dt$$

$$= \frac{m}{2}\left[\dot{\vec{r}}^I(t_B) \cdot \dot{\vec{r}}^I(t_B) - \dot{\vec{r}}^I(t_A) \cdot \dot{\vec{r}}^I(t_A)\right]$$

$$= \frac{1}{2} m V_B^2 - \frac{1}{2} m V_A^2$$

(KINETIC ENERGY)

DENOTE AS: T_B $\quad$ T_A

$$\therefore \quad W\Big|_A^B = T_B - T_A$$

WORK DONE EQUALS THE INCREASE IN KINETIC ENERGY.

- **CONSERVATIVE FORCE**
  - WORK DONE ONLY DEPENDS ON END-POINTS <u>NOT</u> ON THE PATH TAKEN

$$\Rightarrow \quad \vec{F}\cdot d\vec{r} = \underbrace{-dV}$$

↳ - "AN EXACT DIFFERENTIAL"

- $V \sim$ **POTENTIAL ENERGY**

- SO NOW HAVE $$W = \int_A^B \vec{F}\cdot d\vec{r} = -\int_A^B dV$$
$$= V_A - V_B$$

$\Rightarrow$ DECREASE IN POTENTIAL ENERGY IS EQUAL TO THE WORK DONE.

- COMBINE RESULTS

$$W = T_B - T_A = V_A - V_B$$

$$\Rightarrow \quad V_B + T_B = T_A + V_A = E$$

- CALLED THE **PRINCIPLE OF CONSERVATION OF MECHANICAL ENERGY.**
- ONLY APPLIES FOR SYSTEMS IN A **CONSERVATIVE FORCE FIELD.**

- MUCH MORE ON ENERGY METHODS LATER!

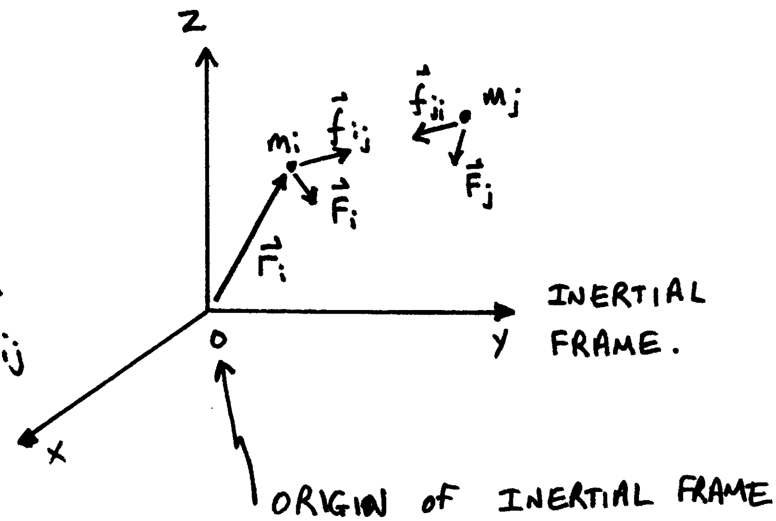
DYNAMICS OF A SYSTEM OF PARTICLES.

- GENERALIZE SINGLE PARTICLE TO MANY
  - STEPPING STONE TO RIGID BODY DYNAMICS

- SIMILAR TO PREVIOUS RESULTS, EXCEPT NOW WE MUST ACCOUNT FOR THE INTERNAL INTERACTIONS OF THE PARTICLES.

- N PARTICLES WITH MASSES m_i

- BOTH INTERNAL $\vec{f}_{ij}$ AND EXTERNAL FORCES $\vec{F}_i$



EXAMPLES OF EACH ?

- NEWTON'S 3RD LAW $\Rightarrow$ $\vec{f}_{ij} = -\vec{f}_{ji}$ $\qquad$ $\vec{f}_{ii} \equiv 0$

- <u>NOTE:</u> $\vec{f}_{ij}$ PARALLEL TO $(\vec{r}_i - \vec{r}_j) \equiv \vec{r}_{ij}$

- WHAT CHANGES WHEN WE GO FROM A SINGLE PARTICLE TO MANY PARTICLES?

  - INTRODUCE CONCEPT OF THE CENTER OF MASS FOR A SYSTEM

    ⇒ FOR BULK PROPERTIES OF THE SYSTEM, CAN JUST TREAT IT AS THE C.O.M. ACTING UNDER THE EXTERNAL FORCES.

  - MOMENTUM, ANGULAR MOMENTUM, FORCES, + TORQUES AND THEIR RELATIONSHIPS DO NOT CHANGE.

  - ENERGY CONCEPTS CAN GET VERY COMPLEX IF THE INTERNAL FORCES ARE NOT CONSERVATIVE.

① MOMENTUM

- MOMENTUM OF MASS i IS: $\vec{p}_i = m_i \dot{\vec{r}}_i^{\,I}$

- TOTAL SYSTEM MOMENTUM IS:

$$\vec{P} = \sum_{i=1}^{N} \vec{p}_i = \sum_{i}^{N} m_i \dot{\vec{r}}_i^{\,I}$$

- ANGULAR MOMENTUM (ABOUT O) IS: $\vec{h}_i = \vec{r}_i \times (m_i \dot{\vec{r}}_i^{\,I})$

- TOTAL SYSTEM ANGULAR MOMENTUM (ABOUT O) IS:

$$\vec{h} = \sum_i \vec{h}_i = \sum_i m_i (\vec{r}_i \times \dot{\vec{r}}_i^{\,I})$$

⇒ NO SURPRISES.

② CENTER OF MASS.

- DEFINED TO BE THE POINT GIVEN BY

$$\vec{r}_c \triangleq \frac{1}{M} \sum_i m_i \vec{r}_i \qquad M = \sum_i m_i$$

- LET $\vec{\rho}_i = \vec{r}_i - \vec{r}_c$

THEN $\sum_i m_i \vec{\rho}_i = 0$ WHY?

⇒ $\sum_i m_i \dot{\vec{\rho}}_i = 0$ RELATIVE TO A FRAME ATTACHED TO THE C.O.M., SYSTEM MOMEMENTUM IS ZERO.

③ FORCES AND TORQUES.

- EQUATION OF MOTION FOR MASS i IS:

$$m_i \ddot{\vec{r}}_i^{\,I} = \vec{F}_i + \sum_j \vec{f}_{ij}$$

- SUM FOR ALL N PARTICLES:

$$\sum_i^N m_i \ddot{\vec{r}}_i^{\,I} = \sum_i^N \vec{F}_i + \sum_i^N \sum_j^N \vec{f}_{ij}$$

WHY?

- BUT WE KNOW THAT $\sum_i^N \sum_j^N \vec{f}_{ij} \equiv 0$

$\Rightarrow$ EQUATIONS OF MOTION FOR SYSTEM ARE:

$$\sum_i m_i \ddot{\vec{r}}_i^{\,I} = \sum_i \vec{F}_i \triangleq \vec{F}$$

TOTAL FORCE ACTING ON THE SYSTEM.

- CAN SIMPLIFY BY NOTING THAT $m \ddot{\vec{r}}_c^{\,I} = \sum_i m_i \ddot{\vec{r}}_i^{\,I}$

$$\Rightarrow \quad \vec{F} = m \ddot{\vec{r}}_c$$

- SO WE CAN TREAT THE MASS CENTER SEPARATELY USING THE EXTERNAL FORCES AND THEN EXPRESS THE MOTION OF EACH PARTICLE WRT THE C.O.M.

MOST IMPORTANT POINT IN THIS LECTURE!

- FOR THE <u>TORQUES</u>, FIRST NOTE THAT SINCE

$$\vec{h} = \sum_i m_i \vec{r}_i \times \dot{\vec{r}}_i^{\,I} \quad \Rightarrow \quad \dot{\vec{h}}^{I} = \sum_i m_i \vec{r}_i \times \ddot{\vec{r}}_i^{\,I}$$

– USE $\quad m_i \ddot{\vec{r}}_i^{\,I} = \vec{F}_i + \sum_j \vec{f}_{ij}$

$$\Rightarrow \dot{\vec{h}}^{I} = \sum_i \vec{r}_i \times \left(\vec{F}_i + \sum_j \vec{f}_{ij} \right)$$

- BUT WE KNOW THAT $\displaystyle \sum_i \sum_j \vec{r}_i \times \vec{f}_{ij} \equiv 0$

WHY?

- THEREFORE WE ARE LEFT WITH:

$$\dot{\vec{h}}^{I} = \sum_i \vec{r}_i \times \vec{F}_i \triangleq \vec{M} \longrightarrow$$

TOTAL TORQUE ON THE SYSTEM ABOUT O.

AGAIN, NO SURPRISES.

- CONVERT TO WORKING ABOUT C.O.M.

$$\vec{r}_i = \vec{r}_c + \vec{\rho}_i$$

$$\vec{h} = \sum_i m_i \vec{r}_i \times \dot{\vec{r}}_i^{\,I} = \sum_i m_i (\vec{r}_c + \vec{\rho}_i) \times (\dot{\vec{r}}_c^{\,I} + \dot{\vec{\rho}}_i^{\,I})$$

$$= \sum_i m_i \vec{\rho}_i \times \dot{\vec{\rho}}_i^{\,I} + \cancelto{0}{\left(\sum_i m_i \vec{\rho}_i\right)} \times \dot{\vec{r}}_c^{\,I} + \vec{r}_c \times \cancelto{0}{\left(\sum_i m_i \dot{\vec{\rho}}_i^{\,I}\right)} + \left(\sum_i m_i\right) \vec{r}_c \times \dot{\vec{r}}_c^{\,I}$$

– SIMPLIFIES SINCE 2 TERMS VANISH.

- DEFINE $\vec{h}_c \equiv \sum_i m_i \vec{\rho}_i \times \dot{\vec{\rho}}_i^{\,I}$ SYSTEM ANGULAR MOM. ABOUT THE C.O.M.

- THEN $\vec{h} = \vec{h}_c + \underbrace{m \vec{r}_c \times \dot{\vec{r}}_c^{\,I}}$

SYSTEM ANGULAR MOM. OF C.O.M. ABOUT O.

- TAKE TIME DERIVATIVE:

$$\dot{\vec{h}}^I = \dot{\vec{h}}_c^I + m\vec{r}_c \times \ddot{\vec{r}}_c^{\,I} \qquad \text{BUT} \quad m\ddot{\vec{r}}_c^{\,I} = \vec{F}$$

$$= \dot{\vec{h}}_c^I + \vec{r}_c \times \vec{F} \qquad (*)$$

- ALREADY STATED THAT $\dot{\vec{h}}^I = \vec{M} = \sum_i \vec{r}_i \times \vec{F}_i$ (5-8)

$\vec{r}_i = \vec{\rho}_i + \vec{r}_c$

$$\dot{\vec{h}}^I = \sum_i \vec{\rho}_i \times \vec{F}_i + \sum_i \vec{r}_c \times \vec{F}_i$$

$$= \sum_i \vec{\rho}_i \times F_i + \vec{r}_c \times \vec{F} \triangleq \vec{M}_c + \vec{r}_c \times \vec{F} \qquad (**)$$

- COMPARE THESE 2 FINAL EQUATIONS: (*) AND (**)

$\dot{\vec{h}}^I = \vec{M}$ AND $\dot{\vec{h}}_c^I = \vec{M}_c$

ABOUT O WRT INERTIAL

ABOUT C.O.M. WRT INERTIAL.

$\Rightarrow \vec{h}$ CONSTANT IF $\vec{M} = 0$.

④ WORK AND KINETIC ENERGY

- IF THE SYSTEM WERE A SINGLE PARTICLE AT THE C.O.M., THE WORK DONE IS:

$$W_c = \int_{\vec{r}_{c_1}}^{\vec{r}_{c_2}} \vec{F} \cdot d\vec{r}_c = \frac{1}{2} m V_{c2}^2 - \frac{1}{2} m V_{c_1}^2$$

WRONG FOR A SYSTEM

- BUT THE SYSTEM IS A COLLECTION OF PARTICLES SO THIS IS NOT THE TOTAL WORK DONE ON IT.

- WORK DONE ON m_i:

$$W_i = \int_{\vec{r}_{i1}}^{\vec{r}_{i2}} \left(\vec{F}_i + \sum_j \vec{f}_{ij} \right) \cdot d\vec{r}_i \qquad \vec{r}_i = \vec{\rho}_i + \vec{r}_c$$

- SUM OVERALL PARTICLES:

$$W = \sum_i W_i = \sum_i \int_1^2 \vec{F}_i \cdot d\vec{r}_c + \sum_i \int_1^2 \vec{F}_i \cdot d\vec{\rho}_i + \cancel{\sum_i \sum_j \int_1^2 \vec{f}_{ij} \cdot d\vec{r}_c}^{\,0}$$

$$+ \sum_i \sum_j \int_1^2 \vec{f}_{ij} \cdot d\vec{\rho}_i$$

$$\Rightarrow \quad W = \int_1^2 \vec{F} \cdot d\vec{r}_c + \sum_i \int_1^2 \left(\vec{F}_i + \sum_j \vec{f}_{ij} \right) \cdot d\vec{\rho}_i$$

TOTAL WORK.

- LAW OF CONSERVATION OF ENERGY APPLIES TO EACH PARTICLE:

$$\Rightarrow W_i = \frac{1}{2} m_i \, \dot{\vec{r}}_i^{\,I} \cdot \dot{\vec{r}}_i^{\,I} \Big|_1^2 \qquad \text{WITH} \qquad \dot{\vec{r}}_i^{\,I} = \dot{\vec{\rho}}_i^{\,I} + \dot{\vec{r}}_c^{\,I}$$

$$= \frac{1}{2} m_i \left(\dot{\vec{\rho}}_i^{\,I} \cdot \dot{\vec{\rho}}_i^{\,I} + 2 \dot{\vec{\rho}}_i^{\,I} \cdot \dot{\vec{r}}_c^{\,I} + \dot{\vec{r}}_c^{\,I} \cdot \dot{\vec{r}}_c^{\,I} \right) \Big|_1^2$$

$$= \frac{1}{2} m_i \left(u_i^2 + 2 \dot{\vec{\rho}}_i^{\,I} \cdot \dot{\vec{r}}_c^{\,I} + v_c^2 \right) \Big|_1^2$$

- SUM OVER ALL PARTICLES:

$$W = \sum_i^N W_i = \frac{1}{2} m v_c^2 \Big|_1^2 + \frac{1}{2} \sum_i^N m_i u_i^2 \Big|_1^2$$

- MIDDLE TERM DROPS OUT, WHY?

∴ DEFINE TOTAL KINETIC ENERGY OF THE SYSTEM AS

$$T = \frac{1}{2} m v_c^2 + \frac{1}{2} \sum_i^N m_i u_i^2 = T_c + \sum_i T_i$$

$$\Rightarrow \quad W = T_2 - T_1$$

- TOTAL KINETIC ENERGY IS EQUAL TO THAT DUE TO:
  (i) TOTAL MASS MOVING WITH VELOCITY OF C.O.M.
  (ii) THE MOTIONS OF EACH PARTICLE RELATIVE TO THE C.O.M.

(5) IF THE EXTERNAL FORCES ARE CONSERVATIVE THEN THE ENERGY

$$E_c = T_c + V_c = \text{CONSTANT.}$$

POTENTIAL ENERGY ASSOCIATED WITH POSITION OF C.O.M.

IF THE INTERNAL FORCES $\vec{f}_{ij}$ ARE ALSO CONSERVATIVE THEN THE TOTAL ENERGY OF THE SYSTEM

$$E = T + V = \text{CONSTANT.}$$

POTENTIAL ENERGY OF ALL PARTICLES.

⇒ READ EXAMPLE 4-1 ON PAGE 141

- <u>CONSERVATIVE FORCE</u> - ONE FOR WHICH

$$\int_A^B \vec{F} \cdot d\vec{r}$$

IS A FUNCTION OF THE END POINTS A AND B, AND INDEPENDENT OF THE PATH TAKEN.