3/3/03
10.537

MACROSCOPIC MATERIAL PROPERTIES

- MOTIVATION

- STRESS & STRAIN

- RELATE STRESS & STRAIN ⇒ HOOKE'S LAW

- MEASURABLES

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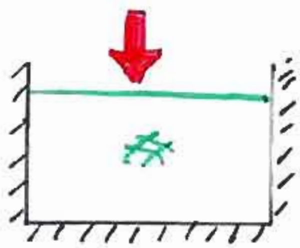
Figure 3, graph of "Tensile stress-strain behavior of N. Clavipes spider silk compared to other textile fibers." In Ko, Frank K., et al. "Engineering Properties of Spider Silk." *MRS Symposium Proceedings*, Vol. 702 (Fall 2001 meetings), paper U1.4.

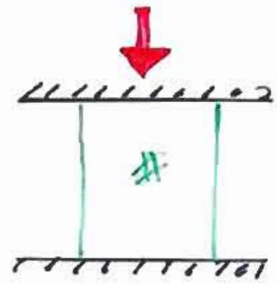
Image removed due to copyright considerations.

"Swelling of Connective Tissues: Confined Compression," shown through graph of stress vs. strain for Bovine Articular Cartilage.

MACROSCOPIC TISSUE-LEVEL BIOMECHANICS

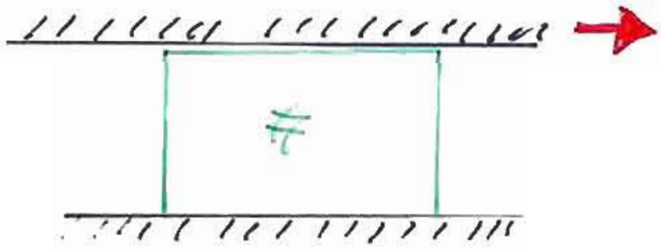
- COMPRESSION





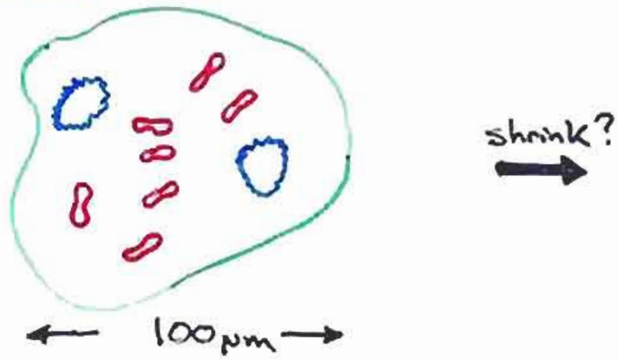
- TENSION

- SHEAR



CONTINUUM MECHANICS?

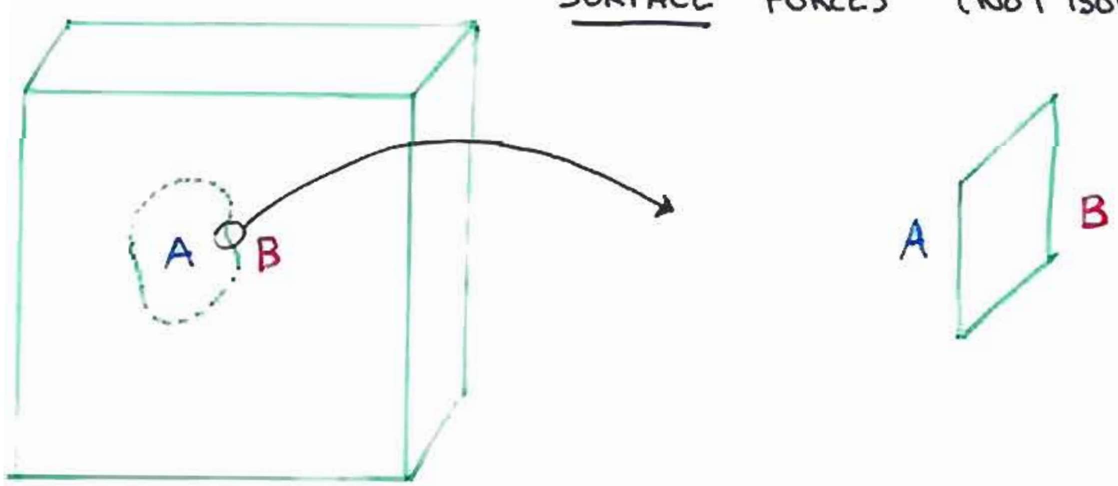
consider blood...



bounds set by physical length scales...
CELLS, MOLECULES ETC..

THE CONCEPT OF STRESS

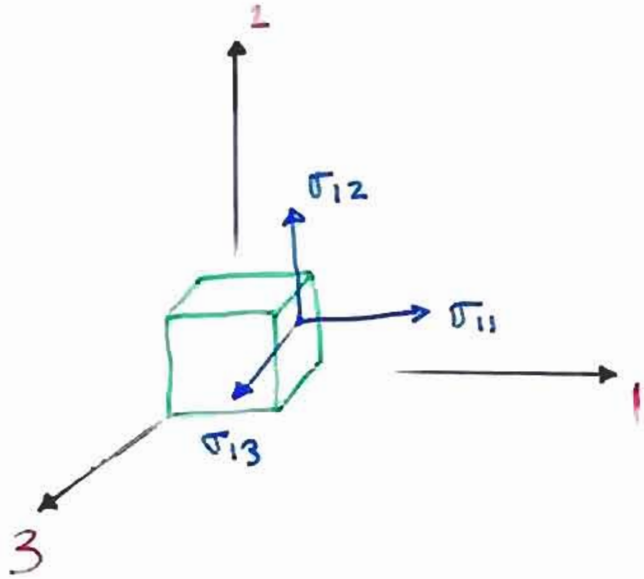
<u>SURFACE</u> FORCES (NOT BODY)



DIMENSIONS: FORCE/AREA

UNITS: N/m^2, Pa

STRESS NOTATION



σ_{ij} STRESS TENSOR

i-component of the force exerted on a plane described by the outward normal in the j-direction.

STRESS VECTOR

$$T_i = \sigma_{ij} n_j$$

FORCE/AREA ON A SURFACE DESCRIBED BY THE NORMAL n_i

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

SHEAR COMPONENTS

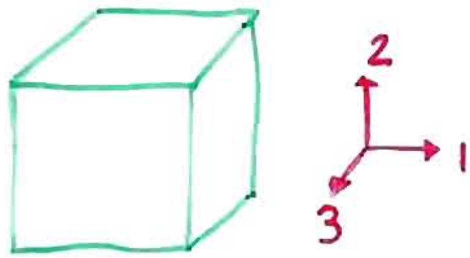
NORMAL COMPONENTS

~~9~~ UNKOWNS

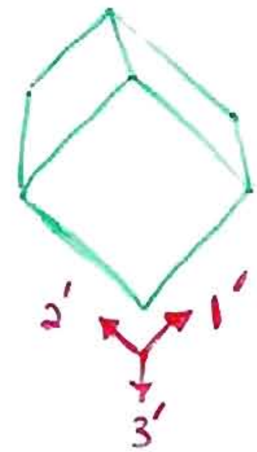
No BODY TORQUES $\Rightarrow$ $\sigma_{ij} = \sigma_{ji}$ symmetric

USING SYMMETRY $\Rightarrow$ 6 UNKOWNS

PRINCIPAL STRESS COMPONENTS



ROTATION OF COORD. SYSTEM



$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

$$\sigma'_{ij} = \begin{bmatrix} \sigma'_{11} & 0 & 0 \\ 0 & \sigma'_{22} & 0 \\ 0 & 0 & \sigma'_{33} \end{bmatrix}$$

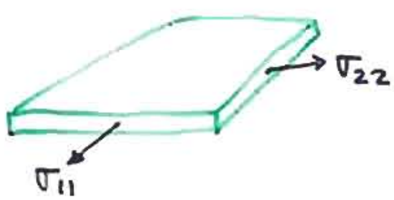
lost info? no.

SHEAR STRESSES = 0

SIMPLE CASES

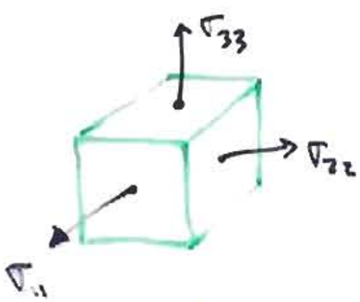
UNIAXIAL TENSION

$$\begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



BIAXIAL TENSION

$$\begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

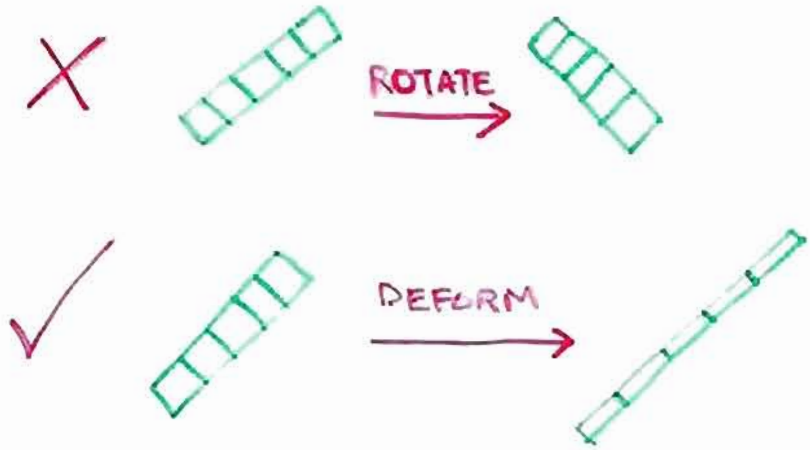


TRIAXIAL TENSION

$$\begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}$$

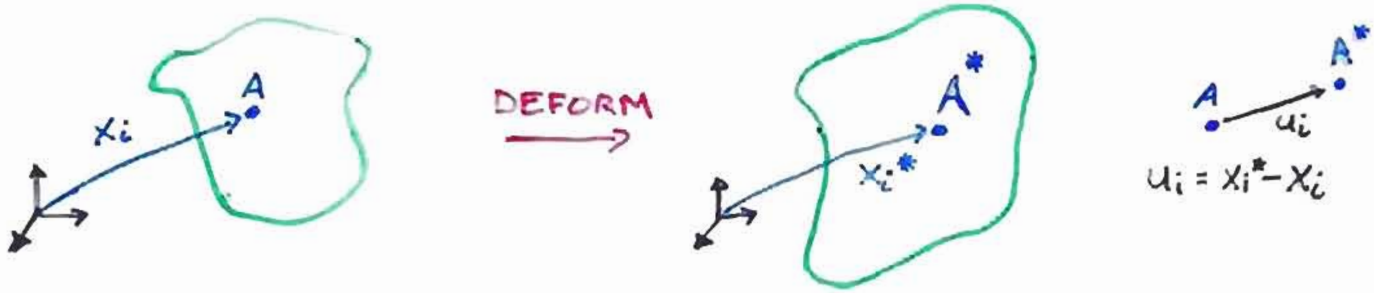
(PRESSURE)

STRAIN



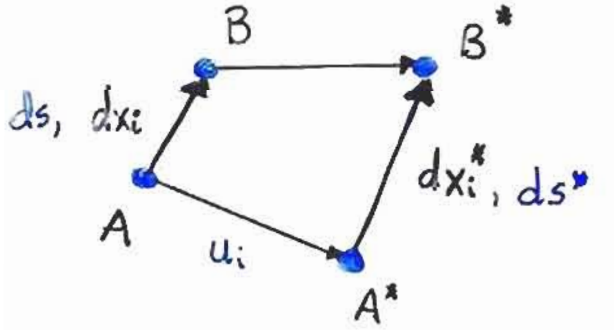
⚠ STRAIN describes deformation (not rigid body rotation or translation) in terms of relative displacements.

MORE GENERALLY:



$u_i = x_i^* - x_i$

NEED 2 POINTS OR CANNOT DISTINGUISH DISPLACEMENT FROM DEFORMATION



'd' denotes a small quantity.

MEASURE OF 'DEFORMATION': $ds^{*2} - ds^2$

$$dx_i^* dx_i^* - dx_i dx_i$$

STRAIN CONT.

$$x_i^* = x_i + u_i$$

$$dx_i^* = \frac{\partial x_i^*}{\partial x_j} dx_j = \frac{\partial u_i}{\partial x_j} dx_j + \underbrace{dx_j \delta_{ij}}_{dx_i}$$

NOW USE IN:

$$dx_i^* dx_i^* - dx_i dx_i = \left(\frac{\partial u_i}{\partial x_j} + \delta_{ij}\right)\left(\frac{\partial u_i}{\partial x_k} + \delta_{ik}\right) dx_j dx_k - dx_i dx_i$$

EXPAND...

$$= \left(\frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j} + \delta_{jk}\right) dx_j dx_k - dx_i dx_i$$

cancel!

$$= \left(\underbrace{\frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_j}}_{\varepsilon^2} + \underbrace{\frac{\partial u_j}{\partial x_k}}_{\varepsilon} + \underbrace{\frac{\partial u_k}{\partial x_j}}_{\varepsilon}\right) dx_j dx_k$$

ASSUME SMALL DEFORMATION...
OK SINCE AFTER A LINEAR THEORY

$$dx_i^* dx_i^* - dx_i dx_i = \left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right) dx_j dx_k$$

$$dx_i^* dx_i^* - dx_i dx_i = 2\,\varepsilon_{jk}\, dx_j dx_k$$

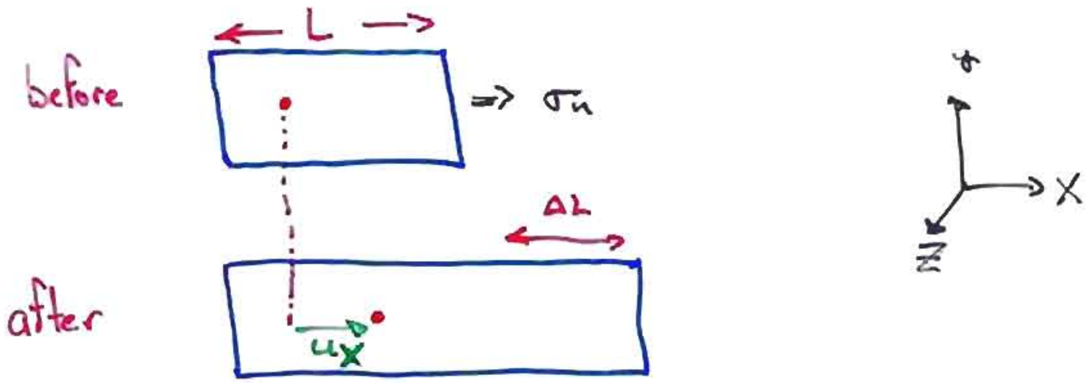
STRAIN TENSOR

$$\varepsilon_{jk} \equiv \frac{1}{2}\left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right)$$

- SYMMETRIC $\varepsilon_{ij} = \varepsilon_{ji}$
- $\varepsilon_{jk} = 0$ FOR SOLID BODY ROTATION

PHYSICAL INTERPRETATION OF STRAIN

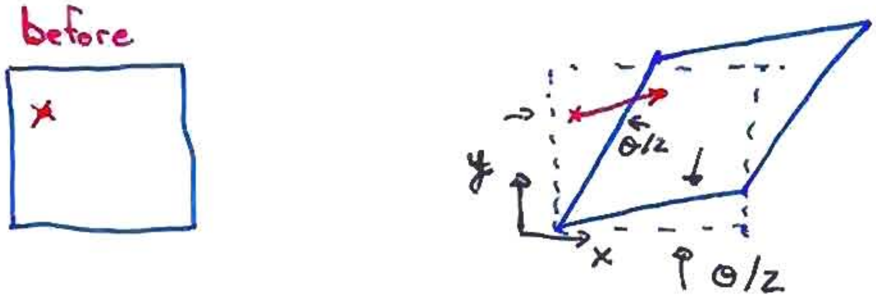
Homogeneous Elongation



$$u_x = \frac{\Delta L}{L} x$$

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = \frac{\Delta L}{L}$$

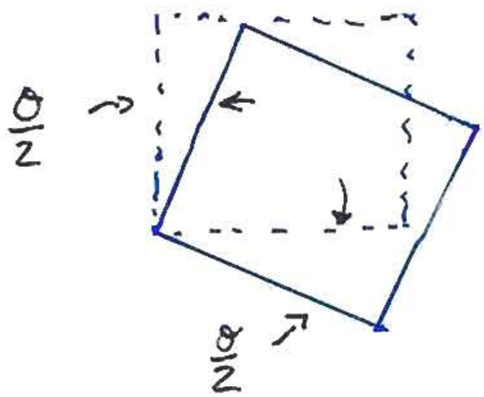
Strain: Shear Deformations



$$\left.\begin{aligned} u_x &\approx \frac{\theta}{2} y \\ u_y &\approx \frac{\theta}{2} x \end{aligned}\right\} \to \varepsilon_{xy} = \varepsilon_{yx} = \frac{1}{2}\left(\frac{du_x}{dy} + \frac{du_y}{dx}\right)$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\theta}{2} + \frac{\theta}{2}\right) = \frac{\theta}{2}$$

Consider Solid Body Rotation:



$$\left.\begin{aligned} u_x &\approx \frac{\theta}{2} y \\ * \quad u_y &\approx -\frac{\theta}{2} x \end{aligned}\right\} \to \varepsilon_{xy} = 0\,!$$

RECALL FLUID DYNAMICS

$$\rho \frac{Dv_i}{Dt} = \rho g_i + \frac{\partial}{\partial x_j} \sigma_{ij}$$

$$\sigma_{ij} = -P\delta_{ij} + \tau_{ij}$$

CONSTITUTIVE RELATION FOR τ_{ij}

postulate

$$\tau_{ij} = C_{ijkl} \underbrace{\frac{\partial \overset{\text{velocity}}{v_k}}{\partial x_l}}_{\text{velocity gradients}}$$

⋮

$$\tau_{ij} = 2\mu E_{ij}$$

NEWTONIAN INCOMPRESSIBLE FLUID

RATE OF STRAIN TENSOR

$$E_{ij} = \frac{1}{2}\left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right]$$

FOR NEWTONIAN FLUIDS THE STRESS DEPENDS ON STRAIN-RATE.

DIMENSIONS:

$$ML^{-1}T^{-2} = [\mu]\, T^{-1}$$

$$[\mu] = ML^{-1}T^{-1}$$

$$[G] = ML^{-1}T^{-2}$$

GENERALIZED HOOKE'S LAW

- CONSTITUTIVE RELATION
- LINEAR ELASTICITY

'COMPLIANCE'

STRESS → $\sigma_{ij} = \underbrace{C_{ijkl}}_{\text{MODULI}} \varepsilon_{kl}$

MODULI (MATERIAL PROPERTIES) → $3^4 = 81$ lots of experiments!

SIMPLIFY : IDEAL & ISOTROPIC MATERIAL

- symmetry of σ_{ij} ⇒ $C_{ijkl} = C_{jikl}$
- symmetry of ε_{kl} ⇒ $C_{ijkl} = C_{ijlk}$
- assume isotropic material ⇒ invariant to coord. trans.

↓

C_{ijkl} 4^{th} order isotropic tensor

GENERAL FORM:

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + G \left(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right)$$

REDUCED TO 2 MODULI (EXPERIMENTS)

λ & G → Lamé (1852) elastic CONSTANTS

DILATION

$$\boxed{\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2 G \varepsilon_{ij}}$$

2nd Lamé CONSTANT

SHEAR MODULUS

$$G = \frac{E}{2(1+\nu)} \quad ; \quad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

ν = POISSON'S RATIO

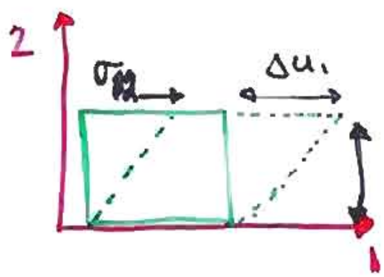
E = YOUNGS MODULUS

⋮

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\,\delta_{ij}$$

PHYSICAL MEANINGS ...

① SHEAR



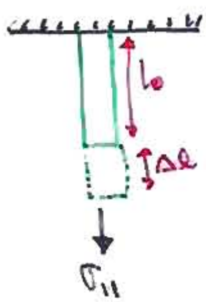
$$\sigma_{12} = 2G\varepsilon_{12}$$

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \underbrace{\cancel{\frac{\partial u_2}{\partial x_1}}}_{0}\right)$$

MEASURE

$$\sigma_{12} = G\frac{\Delta u_1}{l} \approx G\frac{\partial u_1}{\partial x_2}$$

② EXTENSION



$\sigma_{22} = \sigma_{33} = 0$ FREE SURFACE

$$\sigma_{11} / \overset{\approx \Delta l / l_0}{\varepsilon_{11}} = E$$

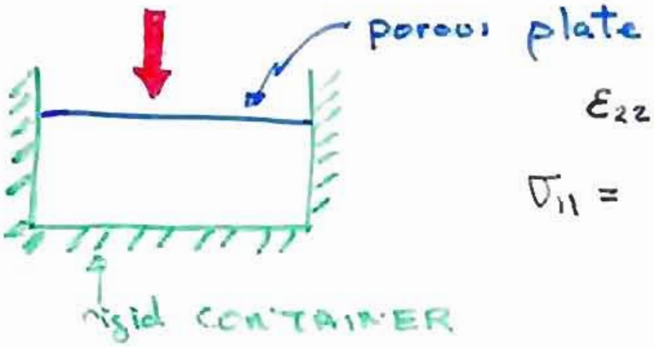
consider ε_{22}

$$\varepsilon_{22} = \frac{1+\nu}{E}\underbrace{\sigma_{22}}_{0} - \frac{\nu}{E}\left(\sigma_{11} + \underbrace{\sigma_{22}}_{0} + \underbrace{\sigma_{33}}_{0}\right)\underbrace{\delta_{22}}_{1}$$

thus $\varepsilon_{22} = -\frac{\nu}{E}\sigma_{11}$; likewise $\varepsilon_{33} = -\frac{\nu}{E}\sigma_{11}$

$$\boxed{\nu = -\frac{\varepsilon_{22}}{\varepsilon_{11}}}$$

③ CONFINED COMPRESSION

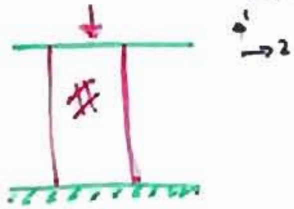


$$\varepsilon_{22} = \varepsilon_{33}$$

$$\sigma_{11} = \underbrace{(2G + \lambda)}_{H} \, \varepsilon_{11}$$

compressive modulus

④ UNCONFINED COMPRESSION



$$\sigma_{22} = \sigma_{33} = 0 \qquad E = \frac{\sigma_{11}}{\varepsilon_{11}}$$

⑤ HYDROSTATIC PRESSURE

$$\sigma_{11} = 2G\,\varepsilon_{11} + \lambda\,\varepsilon_{kk}$$

$$\sigma_{22} = 2G\,\varepsilon_{22} + \lambda\,\varepsilon_{kk}$$

$$\sigma_{33} = 2G\,\varepsilon_{33} + \lambda\,\varepsilon_{kk}$$

$$\frac{\sigma_{ll}}{3} = \underbrace{\left(\lambda + \frac{2}{3}G\right)}_{K} \varepsilon_{kk}$$

negative of pressure

BULK MODULUS

Dilation

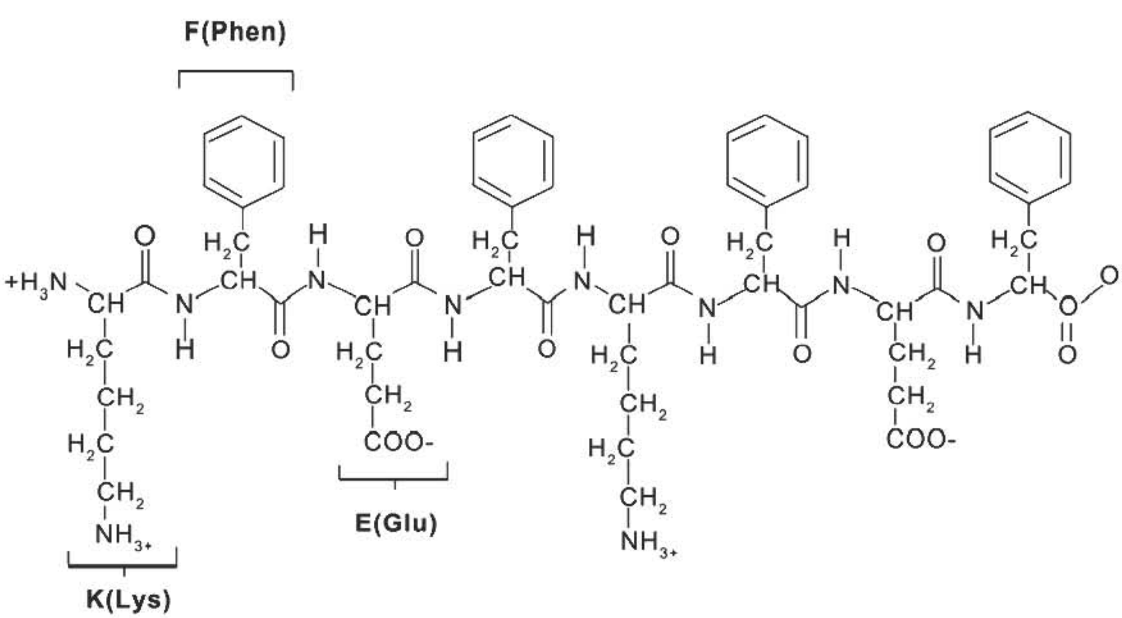
Image removed due to copyright considerations.

"Self Assembling Peptides." Source: Marini, Kamm, et al., 2002.
Four photographs at 8 min, 35 min, 2 hrs, 30 hrs.

Image removed due to copyright considerations.

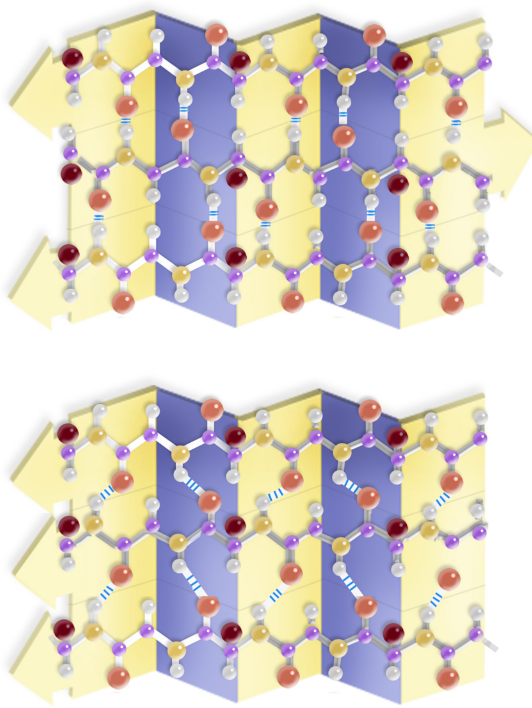
"Self Assembling Peptide Gel." Source: S. Zhang et al.,
Figure 5 (Scanning electron micrographs.

Molecular structure of the oligopeptide, EFK8



Protein Structure and Function

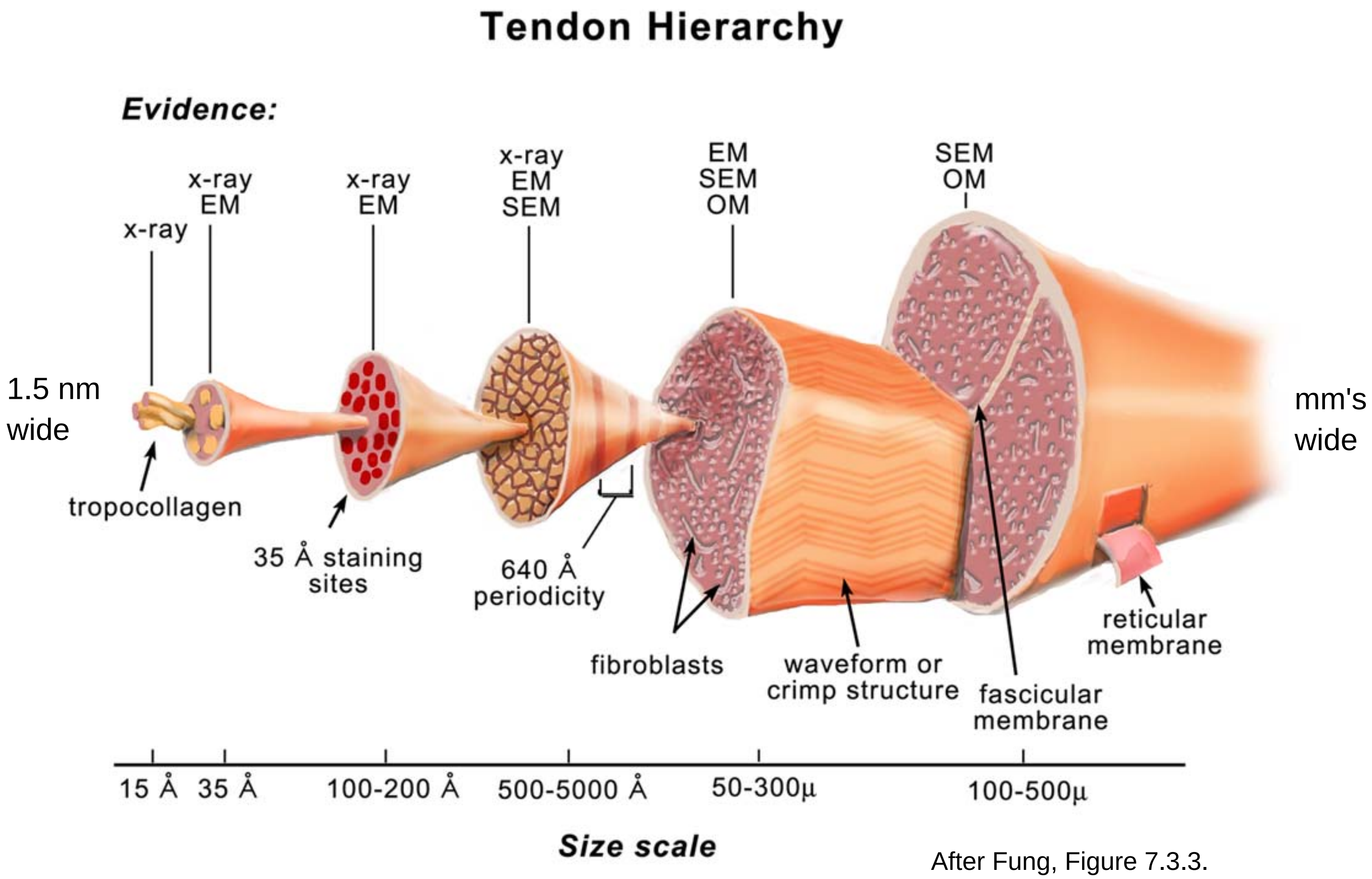
β–sheets



Antiparallel

Parallel

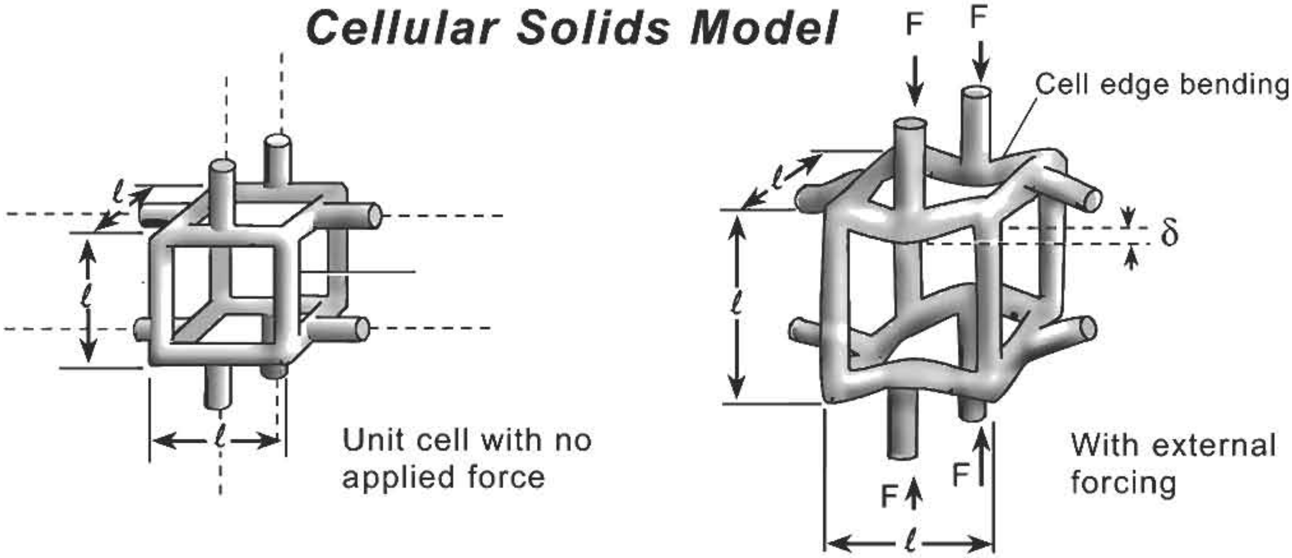
COLLAGEN! Complex Structure Affects Equil. Mechan. Properties Independ. of Fluid Flow



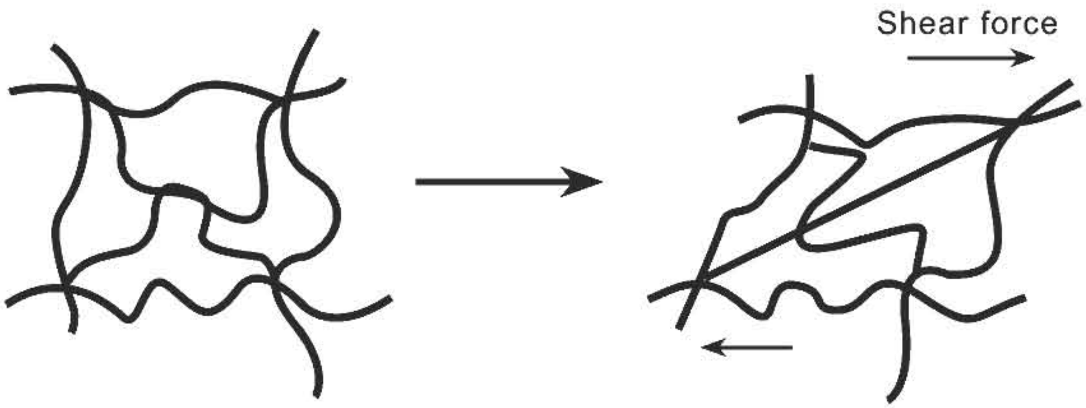
After Fung, Figure 7.3.3.

Images removed due to copyright considerations.
Figure 7.3.4 and 7.3.5 in Fung, Y. C. *Biomechanics: Mechanical Properties of Living Tissues*. New York: Springer-Verlag, 1993.

Cellular Solids Model



Biopolymer Model



SUMMARY

- STRESS & STRAIN

- HOOKE'S LAW $\sigma \sim \varepsilon$
  - ideal, isotropic material $\Rightarrow$ 2 material properties
  - $\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2 G e_{ij}$

    $\varepsilon_{ij} = \frac{1+\nu}{E} \sigma_{ij} - \frac{\nu}{E} \sigma_{kk} \delta_{ij}$

- MEASUREMENTS

  $E,\ G,\ \nu,\ H,\ K$

- SELF ASSEMBLED PEPTIDES