Lecture #12: Looking Backward Before First Hour Exam

Postulates, in the same order as in McQuarrie.

1. $\Psi(r,t)$ is the state function: it tells us everything we are allowed to know
2. For every observable there corresponds a linear, Hermitian Quantum Mechanical operator
3. Any *single* measurement of the property $\widehat{A}$ only gives *one* of the eigenvalues of $\widehat{A}$
4. Expectation values. The average over many measurements on a system that is in a states that is completely specified by a specific $\Psi(x,t)$.
5. TDSE

We will discuss these, and their consequences, in detail now.

Postulate 1.

The state of a Quantum Mechanical system is *completely* specified by $\Psi(\mathbf{r},t)$

* $\Psi^*\Psi dxdydz$ is the probability that the particle lies within the volume element $dxdydz$ that is centered at

$$\vec{\mathbf{r}} = x\hat{i} + y\hat{j} + z\hat{k} \qquad (\hat{i}, \hat{j}, \text{ and } \hat{k} \text{ are unit vectors})$$

* Ψ is "well behaved"
normalizable (in either of two senses: what are these two senses?)
square integrable [usually requires that $\lim_{x\to\pm\infty} \psi(x) \to 0$]

$$\left.\begin{cases}\text{continuous}\\ \text{single-valued}\\ \text{finite everywhere}\end{cases}\right\} \psi \text{ and } \frac{d\psi}{dx}$$

When do we get to break some of the rules about "well behaved"? (from non-physical but illustrative problems)?

*A finite step in V(x) causes discontinuity in $\dfrac{\partial^2\psi}{\partial x^2}$

*A δ-function (infinite sharp spike) and infinite step in V(x) cause a discontinuity in $\dfrac{\partial\psi}{\partial x}$

Nothing can cause a discontinuity in ψ.
When $V(x) = \infty$, $\psi(x) = 0$. Always! [Why?]

Postulate 2

For every observable quantity in Classical Mechanics there corresponds a linear, Hermitian Operator in Quantum Mechanics.

linear means $\widehat{A}\left(c_1\psi_1 + c_2\psi_2\right) = c_1\widehat{A}\psi_1 + c_2\widehat{A}\psi_2$. We have already discussed this.

Hermitian is a property that ensures that every observation results in a *real* number (not imaginary, not complex)

A Hermitian operator satisfies

$$\int_{-\infty}^{\infty} f^*(\hat{A}g)\,dx = \int_{-\infty}^{\infty} g(\hat{A}^* f^*)\,dx$$

$$A_{fg} = \left(A_{gf}\right)^* \quad \text{(useful short-hand notation)}$$

where *f* and *g* are well-behaved functions.

This provides a very useful prescription for how to "operate to the left".

Suppose we replace *g* by *f* to see how Hermiticity ensures that any measurement of an observable quantity must be real.

$$\int_{-\infty}^{\infty} f^*\widehat{A}f dx = \int_{-\infty}^{\infty} f\widehat{A}^* f^* dx \text{ from the definition of Hermitian}$$

$$A_{ff} = (A_{ff})^*$$

The LHS is just $\langle\widehat{A}\rangle_f$, the expectation value of $\widehat{A}$ in state f.

The RHS is just LHS*, which means

$$\text{LHS} = \text{LHS}^*$$

thus $\langle\widehat{A}\rangle_f$ is real.

Non-Lecture

Often, to construct a Hermitian operator from a non-Hermitian operator, $\hat{A}_{\text{non-Hermitian}}$, we take

$$\widehat{A}_{\text{QM}} = \frac{1}{2}\left(\widehat{A}_{\text{non-Hermitian}} + \widehat{A}^*_{\text{non-Hermitian}}\right).$$

OR, when an operator $\widehat{C} = \widehat{A}\widehat{B}$ is constructed out of non-commuting factors, e.g.

$$\left[\hat{A},\hat{B}\right] \neq 0 .$$

Then we might try $\hat{C}_{\text{Hermitian}} = \frac{1}{2}\left(\hat{A}\hat{B} + \hat{B}\hat{A}\right)$.

<u>Angular Momentum</u>

Classically

$$\vec{\ell} = \hat{r} \times \hat{p} = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{pmatrix}$$

$$\boxed{\ell_x \hat{i} + \ell_y \hat{j} + \ell_z \hat{k}}$$

$\ell_x = y p_z - z p_y$ Does order matter?

$$\left.\begin{matrix} \left[y, p_z\right] = 0 \\ \left[z, p_y\right] = 0 \end{matrix}\right) \text{ by inspection (of what?)}$$

which is a good thing because the standard way for compensating for non-commutation,

$$\hat{r} \times \hat{p} + \hat{p} \times \hat{r} = 0$$

fails, so we would not be able to guarantee Hermiticity this way

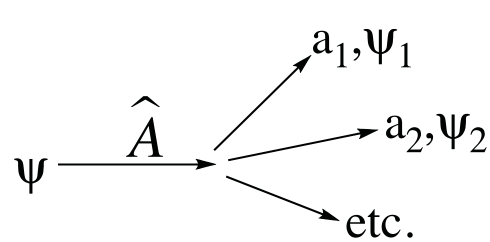
End of Non-Lecture

<u>Postulate 3</u>

Each measurement of the observable quantity associated with $\hat{A}$ gives one of the eigenvalues of $\hat{A}$.

$\hat{A}\psi_n = a_n \psi_n$ the set of all eigenvalues, $\{a_n\}$, is called **<u>spectrum</u>** of $\hat{A}$

Measurements:



Measurement causes an arbitrary ψ to "collapse" into one of the eigenstates of the measurement operator.

Postulate 4

For a system in *any* state normalized to 1, ψ, the average value of $\hat{A}$ is $\langle \hat{A} \rangle \equiv \int_{-\infty}^{\infty} \psi^* \hat{A} \psi d\tau$.

(dτ means integrate over all coordinates).

We can combine postulates 3 and 4 to get some very useful results.

1. Completeness (with respect to each operator)

$\psi = \sum_i c_i \psi_i$ expand ψ in a "complete basis set" of eigenfunctions, ψ_i

(many choices of "basis sets")

Most convenient to use all eigenstates of $\hat{A}$ $\{\psi_i\},\{a_i\}$
We often use a complete set of eigenstates of $\hat{A}\{\psi_n^A\}$ as "basis states" for the operator $\hat{B}$ even when the $\{\psi_n^A\}$ are *not eigenstates* of $\hat{B}$.

2. Orthogonality

If ψ_i,ψ_j belong to $a_i \neq a_j$, then $\int dx \psi_i^* \psi_j = 0$. Even when we have a *degenerate* eigenvalue, where $a_i = a_j$, we can construct orthogonal functions. For example:

$\hat{A}\psi_1 = a_1\psi_1$, $\hat{A}\psi_2 = a_1\psi_2$, ψ_1,ψ_2 are normalized but not necessarily orthogonal.

NON-Lecture

Construct a pair of normalized and orthogonal functions starting from ψ_1 and ψ_2.

Schmidt orthogonalization

$$
\begin{aligned}
S &\equiv \int dx \psi_1^* \psi_2 \neq 0, \text{ the overlap integral} \\
\psi_2' &= N\left(\psi_2 + a\psi_1\right), \text{ constructed to be orthogonal to } \psi_1 \\
\int dx \psi_1^* \psi_2' &= N \int dx \psi_1^* \left(\psi_2 + a\psi_1\right) \\
&= N\left(S + a\right).
\end{aligned}
$$

If we set $a = -S$, ψ'_2 is orthogonal to ψ_1. We must normalize ψ'_2.

$$
\begin{aligned}
1 &= \int dx \psi_2'^* \psi_2' = |N|^2 \int dx \left(\psi_2^* - S^* \psi_1^*\right)\left(\psi_2 - S\psi_1\right) \\
&= |N|^2 \left[1 - 2|S|^2 + |S|^2\right] \\
N &= \left[1 - |S|^2\right]^{-1/2} \\
\psi_2' &= \left[1 - |S|^2\right]^{-1/2} \left(\psi_2 - S\psi_1\right)
\end{aligned}
$$

ψ'_2 is normalized to 1 and orthogonal to ψ_1. This turns out to be a *very* useful trick.

"Complete orthonormal basis sets"

Next we want to compute the $\{c_i\}$ and the $\{P_i\}$. P_i is the probability that an experiment on ψ yields the i^{th} eigenvalue.

$$\psi = \sum_i c_i \psi_i$$

(ψ is any normalized state)

Left multiply and integrate by ψ_j^* (which is the complex conjugate of the eigenstate of $\hat{A}$ that belongs to eigenvalue a_j).

$$
\begin{aligned}
\int dx \psi_j^* \psi &= \int dx \psi_j^* \sum_i c_i \psi_i \\
&= \sum_i c_i \delta_{ji} \\
c_j &= \int dx \psi_j^* \psi \text{ (so we can compute all } \{c_i\})
\end{aligned}
$$

What about

$$
\begin{aligned}
\langle \hat{A} \rangle &= \sum_i P_i a_i \\
\int dx \psi^* \hat{A} \psi &= \int dx \left[\sum_i c_i^* \psi_i^*\right] \hat{A} \left[\sum_j c_j \psi_j\right] \\
&= \int dx \left[\sum_i c_i^* \psi_i^*\right]\left[\sum_j a_j c_j \psi_j\right]
\end{aligned}
$$

Orthonormality kills all terms
in the sum over j except $j = i$.

$$\int dx\psi^* \hat{A}\psi = \sum_i |c_i|^2 a_i$$

thus $\langle \hat{A} \rangle = \sum_i |c_i|^2 a_i$

$$P_i = |c_i|^2 = \left| \int dx \psi_i^* \psi \right|^2$$

so the "mixing coefficients" in ψ

$$\psi = \sum c_i \psi_i$$

become "fractional probabilities" in the results of repeated measurements of **A.**

$$\langle \hat{A} \rangle = \sum P_i a_i$$

$$P_i = \left| \int dx \psi_i^* \psi \right|^2 .$$

What does the $\left[\hat{A},\hat{B}\right]$ commutator tell us about

* the possibility for simultaneous eigenfunctions
* $\sigma_A \sigma_B$?

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1. If $\left[\hat{A},\hat{B}\right] = 0$, then all non-degenerate eigenfunctions of $\hat{A}$ are eigenfunctions of $\hat{B}$ (see page 10).

2. If $\left[\hat{A},\hat{B}\right] = \text{const} \neq 0$

$$\sigma_A^2 \sigma_B^2 \geq -\frac{1}{4}\left(\int dx\psi^*[A,B]\psi\right)^2 > 0 \text{ (and real)}$$

note that $\left[\hat{x},\hat{p}\right] = i\hbar$

this gives

$$\sigma_{p_x} \sigma_x \geq \frac{\hbar}{2}$$ (see page 11)

NON-LECTURE

Suppose 2 operators commute

$$\left[\hat{A},\hat{B}\right] = 0$$

Consider the set of wavefunctions $\{\psi_i\}$ that are eigenfunctions of observable quantity $\widehat{A}$.

$$\widehat{A}\psi_i = a_i\psi_i \qquad \{a_i\} \text{ are real}$$

$$\underbrace{0}_{\text{commutator is 0}} = \int dx\psi_j^*\left[\widehat{A},\widehat{B}\right]\psi_i = \int dx\psi_j^*\left(\widehat{A}\widehat{B} - \widehat{B}\widehat{A}\right)\psi_i$$

$$= \int dx\psi_j^*\widehat{A}\widehat{B}\psi_i - \int dx\psi_j^*\widehat{B}\widehat{A}\psi_i$$

$$= a_j\int dx\psi_j^*\widehat{B}\psi_i - a_i\int dx\psi_j^*\widehat{B}\psi_i$$

$$= \left(a_j - a_i\right)\int dx\psi_j^*\widehat{B}\psi_i$$

$$0 = \left(a_j - a_i\right)\underbrace{\int dx\psi_j\widehat{B}\psi_i}_{B_{ji}}$$

if $a_j \neq a_i \rightarrow B_{ji} = 0$ — this implies that ψ_i and ψ_j are eigenfunctions of $\hat{B}$ that belong to different eigenvalues of $\hat{B}$

if $a_j = a_i \rightarrow B_{ji} \neq 0$ — This implies that we can construct mutually orthogonal eigenfunctions of $\hat{B}$ from the set of degenerate eigenfunctions of $\hat{A}$.

All nondegenerate eigenfunctions of $\widehat{A}$ are eigenfunctions of $\widehat{B}$ and eigenfunctions of $\widehat{B}$ can be constructed out of degenerate eigenfunctions of $\widehat{A}$.

Some important topics:

0. Completeness.
1. For a Hermitian Operator, all non-degenerate eigenfunctions are orthogonal and the non-degenerate ones can be made to be orthonormal.
2. Schmidt orthogonalization
3. Are eigenfunctions of $\widehat{A}$ eigenfunctions of $\widehat{B}$ if $\left[\widehat{A},\widehat{B}\right] = 0$?
4. $\left[\widehat{A},\widehat{B}\right] \neq 0 \Rightarrow$ uncertainty principle free of any thought experiments.
5. Why do we define $\hat{p}$ as $-i\hbar\dfrac{\partial}{\partial x}$?
6. Express non-eigenstate as linear combination of eigenstates.

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0. <u>Completeness.</u> Any arbitrary ψ can be expressed as a linear combination of functions that are members of a "complete basis set."

For a particle in box

$$\psi_n = \left(\frac{2}{a}\right)^{1/2} \sin\left(\frac{n\pi}{a}x\right)$$

$$E_n = n^2 \frac{h^2}{8ma^2}$$

complete set n = 1, 2, … ∞ What do we call these ψ_n in a non-QM context?

$$\psi = \sum_i c_i \psi_i, \quad c_i = \int dx \psi_i^* \psi$$

To obtain the set of $\{c_i\}$, left-multiply ψ by Ψ_i^* and integrate. Exploit orthonormality of the basis set $\{\psi_i\}$.

<u>Fourier series</u>: any arbitrary, well-behaved function, defined on a finite interval $(0,a)$, can be decomposed into orthonormal Fourier components.

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{a} + b_n \sin\frac{n\pi x}{a}\right).$$

For our usual $\psi(0) = \psi(a) = 0$ boundary conditions, all of the $a_n = 0$. We can use particle in box functions $\{\psi_n\}$ to express any ψ where $\psi(0) = \psi(a) = 0$. Another kind of boundary condition is periodic (e.g. particle on a ring) $\psi(x + a) = \psi(x)$ where a is the circumference of the ring. Then, for the $0 \le x \le a$ interval, we need both sine and cosine Fourier series.

1. <u>Hermitian Operator</u>

If $\hat{A}$ is Hermitian, all of the non-degenerate eigenstates of $\hat{A}$ are orthogonal and all of the degenerate ones can be made orthogonal.

If $\hat{A}$ is Hermitian

$$\int dx \psi_i^* \underbrace{\widehat{A}\psi_j}_{a_j\psi_j} = \int dx \psi_j \underbrace{\widehat{A}^*\psi_i^*}_{a_i^*\psi_i^*}$$

$\boxed{a_i^* = a_i \text{ because } \widehat{A} \text{ corresponds to a classically observable quantity}}$

rearrange

$$\left(a_j - a_i\right)\int dx \quad \psi_i^* \psi_j \quad = 0$$

order of these doesn't matter

either $a_j = a_i$ (degenerate eigenvalue)

OR

when $a_j \neq a_i$ ψ_i is orthogonal to ψ_j.

Now, when ψ_i and ψ_j belong to a degenerate eigenvalue, they can be made to be orthogonal, yet remain eigenfunctions of $\widehat{A}$.

$$\widehat{A}\left(\sum_i c_i \psi_i\right) = a_j \left(\sum_i c_i \psi_i\right)$$

for any linear combination of degenerate eigenfunctions.

Find the correct linear combination. Easy to get a computer to find these orthogonalized functions.

Non-Lecture

2. <u>Schmidt orthogonalization</u>

We can construct a set of mutually orthogonal functions out of a set of non-orthogonal degenerate eigenfunctions.

Consider two-fold degenerate eigenvalue a_1 with non-orthogonal eigenfunctions, ψ_{11} and ψ_{12}.

Construct a new pair of orthogonal eigenfunctions that belong to eigenvalue a_1 of $\hat{A}$.

$$\text{overlap } S_{11,12} = \int \psi_{11}^{*}\psi_{12}$$
$$\psi'_{11} \equiv \psi_{11}$$
$$\psi'_{12} \equiv N\left[\psi_{12} - S_{11,12}\psi_{11}\right]$$

Check for orthogonality:

$$\int dx \psi_{11}'^{*}\psi'_{12} = N\left[\int dx \psi_{11}^{*}\psi_{12} - S_{11,12}\int dx \psi_{11}^{*}\psi_{11}\right]$$
$$= N\left[S_{11,12} - S_{11,12}\right] = 0.$$

Find normalization constant:

$$1 = \int dx \psi_{12}'^{*}\psi'_{12}$$
$$= |N|^2 \left[\begin{array}{l} \int dx \psi_{12}^{*}\psi_{12} + \left|S_{11,12}\right|^2 \int dx \psi_{11}^{*}\psi_{11} \\ \qquad - \int dx \psi_{12}^{*} S_{11,12}\psi_{11} - \int dx S_{11,12}^{*}\psi_{11}^{*}\psi_{12} \end{array}\right]$$
$$= |N|^2 \left[1 + \left|S_{11,12}\right|^2 - \left|S_{11,12}\right|^2 - \left|S_{11,12}\right|^2\right]$$
$$= |N|^2 \left[1 - \left|S_{11,12}\right|^2\right]$$
$$N = \left[1 - \left|S_{11,12}\right|^2\right]^{-1/2}$$
$$\psi'_{12} = \left[1 - \left|S_{11,12}\right|^2\right]^{-1/2}\left[\psi_{12} - S_{11,12}\psi_{11}\right]$$

Now we have a complete set of orthonormal eigenfunctions of $\hat{A}$. Extremely convenient and useful.

End of Non-Lecture

3. Are eigenfunctions of $\hat{A}$ also eigenfunctions of $\hat{B}$ if $\left[\hat{A},\hat{B}\right] = 0$?

$$\hat{A}\hat{B} = \hat{B}\hat{A}$$
$$\hat{A}\left(\hat{B}\psi_i\right) = \hat{B}\left(\hat{A}\psi_i\right) = a_i\left(\hat{B}\psi_i\right)$$

thus $\hat{B}\psi_i$ is eigenfunction of $\hat{A}$ belonging to eigenvalue a_i. If a_i is non-degenerate, $\hat{B}\psi_i = c\psi_i$, thus ψ_i is also an eigenfunction of $\hat{B}$.

We can arrange for one set of functions $\{\psi_i\}$ to be simultaneously eigenfunctions of $\hat{A}$ and $\hat{B}$ when $\left[\hat{A},\hat{B}\right]=0$.

This is very convenient. For example: n_x, n_y, n_z for 3D box and eigenvalues of $\widehat{J^2}$ and $\widehat{J_z}$ for rigid rotor. Another example: 1D box has non-degenerate eigenvalues. Thus every eigenstate of $\hat{H}$ is an eigenstate of a symmetry operator that commutes with $\hat{H}$.

4. $\left[\hat{A},\hat{B}\right]\neq 0 \Rightarrow$ uncertainty principle free of any thought expt.

Suppose 2 operators do not commute

$$\left[\hat{A},\hat{B}\right]=\hat{C}\neq 0.$$

It is possible (we will not do it) to prove, for any Quantum Mechanical state ψ

$$\sigma_A^2\sigma_B^2 \geq -\frac{1}{4}\left(\int dx\psi^*\hat{C}\psi\right)^2 \geq 0.$$

Consider a specific example:

$$\hat{A}=\hat{x}$$

$$\hat{B}=\hat{p}_x$$

$$
\begin{aligned}
\left[\hat{x},\hat{p}_x\right]f(x) &= \hat{x}\hat{p}_x f - \hat{p}_x\hat{x}f \\
&= x(-i\hbar)\frac{\partial}{\partial x}f - (-i\hbar)\frac{\partial}{\partial x}(xf) \\
&= (-i\hbar)\left[xf' - f - xf'\right] \\
&= +i\hbar f \\
\therefore \quad \left[\hat{x},\hat{p}_x\right] &= +i\hbar \underbrace{\hat{I}}_{\substack{\Downarrow \\ \text{unit} \\ \text{operator}}}
\end{aligned}
$$

so the above (unproved) theorem says

$$\sigma_x^2\sigma_{p_x}^2 \geq -\frac{1}{4}\left[i\hbar \underbrace{\int dx\psi^*\psi}_{=1}\right]^2 = -(-1)\frac{\hbar^2}{4}$$

$$\sigma_x\sigma_p \geq +\frac{\hbar}{2} \qquad \text{Heisenberg uncertainty principle}$$

This is better than a thought experiment because it comes from the mathematical properties of operators rather than being based on how good one's imagination is in defining an experiment to measure x and p_x simultaneously.

Non-Lecture

5. Why do we define $\hat{p}$ as $\hat{p} = -i\hbar\frac{\partial}{\partial x}$?

Is the -i needed? Why not +i?

$$\langle\hat{p}\rangle = -i\hbar\int_{-\infty}^{\infty} dx\psi^*\frac{d}{dx}\psi$$

which must be real, $\langle\hat{p}\rangle = \langle\hat{p}\rangle^*$. But is it?

integrate by parts,
treat ψ^* and ψ as
linearly independent
functions

$$\langle p\rangle^* = +i\hbar\int_{-\infty}^{\infty} dx\,\psi\frac{d}{dx}\psi^* = +(i\hbar)\left[\underbrace{\psi\psi^*\Big|_{-\infty}^{\infty}}_{0} - \int dx\frac{d\psi}{dx}\psi^*\right] = \langle p\rangle$$

took complex
conjugate of the
equation for $\langle p\rangle$

because ψ,ψ^* must
go to zero at $\pm\infty$

thus $\langle p\rangle = \langle p\rangle^*$, i is needed in $\hat{p}$.

i vs. $-i$ is an arbitrary phase choice, supported by a physical argument.

Suppose we have

$$\psi = e^{ikx}$$

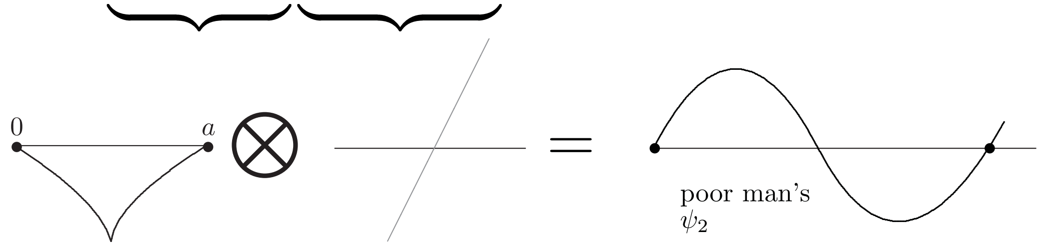
$$\hat{p}\psi = -i\hbar(ik)e^{ikx} = +\hbar k e^{ikx}$$

we like to associate $\langle\hat{p}\rangle$ with $+\hbar k$ rather than $-\hbar k$.

6. <u>Suppose we have a non-eigenstate ψ for the particle in a box</u>

for example,

$$\psi(x) = N\underbrace{x(x-a)}\,\underbrace{(x-a/2)}$$



Normalize this

$$\int_0^a dx\ \psi^*\psi = 1 = N^2 \int_0^a dx\ x^2 (x-a)^2 (x-a/2)^2$$

find that $N = \left(\frac{840}{a^7}\right)^{1/2}$.

Now expand this function in the $\psi_n = \left(\frac{2}{a}\right)^{1/2} \sin\frac{n\pi x}{a}$ basis set.

$$\psi = \sum_{n=1}^{\infty} c_n \psi_n$$ find the c_n

Left multiply by ψ_m^* and integrate

$$\int dx \psi_m^* \psi = \sum_{n=1}^{\infty} c_n \int dx \underbrace{\psi_m^* \psi_n}_{\text{orthogonal}} = c_m$$

$$c_m = (840)^{1/2} a^{-7/2} \left(\frac{2}{a}\right)^{1/2} \int_0^a dx \underbrace{x(x-a)(x-a/2)}_{\text{odd with respect to } 0,a \text{ interval}} \sin \underbrace{\frac{m\pi x}{a}}_{\text{needs to be odd on } 0,a \text{ too in order to have an even integrand}}$$

thus $c_m = 0$ for all odd-m

$m = 2n-1 \qquad n = 1,2,\ldots$
$c_{2n-1} = 0$
$c_{2n} \neq 0$ find them

$$c_{2n} = \frac{(1680)^{1/2}}{a^4}\int_0^a dx\left(x^3 - \frac{3}{2}ax^2 + \frac{a^2}{2}x\right)\sin\frac{2n\pi x}{a}$$

change variables $y = \dfrac{2n\pi x}{a}$

$$= \frac{1680^{1/2}}{a^4}\int_0^{2n\pi} dy\left[\left(\frac{a}{2n\pi}\right)^3 y^3 - \frac{3}{2}a\left(\frac{a}{2n\pi}\right)^2 y^2 + \frac{a^2}{2}\left(\frac{a}{2n\pi}\right)y\right]\left(\frac{a}{2n\pi}\right)\sin y$$

steps skipped

$$c_{2n} = 1680^{1/2}\frac{6}{(2n\pi)^3} = 0.9914\ n^{-3}$$

$c_2 \approx 1$ as expected from general shape of ψ.

Now that we have $\{c_n\}$, we can compute $\langle E\rangle = \int dx\ \psi^* \widehat{H}\psi = \sum_{n=1}^{\infty} \underbrace{P_n}_{\text{prob-ability}} E_n$

$P_n = c_n^2$

$$\langle E\rangle = \sum_{n=1} E_{2n}\left|c_{2n}\right|^2 = E_1\sum_{n=1}^{\infty}(2n)^2\left[0.9914n^{-3}\right]^2$$

$$= 4E_1(0.983)\sum_{n=1}^{\infty} n^{-4} \approx 4E_1$$

(Is this a surprise for a function constructed to resemble ψ_2 where E_2 = $4E_1$?)

End of Non-Lecture