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1.010 Uncertainty in Engineering  
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## Application Example 1

(Probability of combinations of events; binomial and Poisson distributions)

# RELIABILITY OF SYSTEMS WITH VARIOUS ELEMENT CONFIGURATIONS

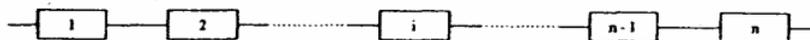
*Note: Sections 1, 3 and 4 of this application example require only knowledge of events and their probability. Section 2 involves the binomial and Poisson distributions.*

### 1: SERIES AND PARALLEL SYSTEMS

Many physical and non-physical systems (e.g. bridges, car engines, air-conditioning systems, biological and ecological systems, chains of command in civilian or military organizations, quality control systems in manufacturing plants, etc.) may be viewed as assemblies of many interacting elements. The elements are often arranged in mechanical or logical series or parallel configurations.

#### *Series systems*

Series systems function properly only when all their components function properly. Examples are chains made out of links, highways that may be closed to traffic due to accidents at different locations, the food chains of certain animal species, and layered company organizations in which information is passed from one hierarchical level to the next.



*Schematic illustration of a series system*

The reliability of a series system is easily calculated from the reliability of its components. Let  $Y_i$  be an indicator of whether component  $i$  fails or not; hence  $Y_i = 1$  if component  $i$  fails and  $Y_i = 0$  if component  $i$  functions properly. Also denote by  $P_i = P[Y_i = 1]$  the probability that component  $i$  fails. The probability of failure of a system with  $n$  components in series is then

$$\begin{aligned} P[\text{system failure}] &= 1 - P[\text{system survival}] \\ &= 1 - P[(Y_1 = 0) \cap (Y_2 = 0) \cap \dots \cap (Y_n = 0)] \end{aligned} \quad (1)$$

If the components fail or survive independently of one another, then this probability becomes

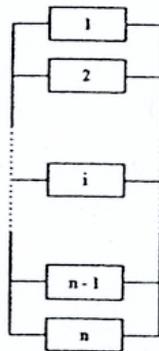
$$P[\text{system failure}] = 1 - \prod_{i=1}^n (1 - P_i) \quad (2)$$

In the even more special case when the component reliabilities are all the same,  $P_i = P$  and Eq. 2 gives

$$P[\text{system failure}] = 1 - (1 - P)^n \quad (3)$$

### ***Parallel systems***

In this case, the system fails only if all its components fail. For example, if an office has  $n$  copy machines, it is possible to copy a document if at least one machine is in good working conditions.



*Schematic illustration of a parallel system*

The probability of failure of a parallel system of this type is obtained as

$$\begin{aligned}
 P[\text{system failure}] &= P[(Y_1 = 1) \cap (Y_2 = 1) \cap \dots \cap (Y_n = 1)] \\
 &= \prod_{i=1}^n P_i, \quad \text{if the components fail independently} \quad (4) \\
 &= P^n, \quad \text{if in addition } P_i = P \text{ for all } i
 \end{aligned}$$

**Problem 1.1** Consider a series system. Plot its probability of failure in Eq. 3 as a function of the number of components  $n$ , for different values of  $P$ . Do the same for parallel systems, using the last expression in Eq. 4. Comment on the effect of  $n$  in the two cases.

## 2: m-out-of-n SYSTEMS

Simple series and parallel representations are often inadequate to describe real systems. A first generalization, which includes series and parallel systems as extreme cases, is that of “m-out-of-n” systems. These systems fail if  $m$  or more out of  $n$  components fail. The case  $m = 1$  corresponds to series systems, the case  $m = n$  to parallel systems. Again, analysis is simpler if the components fail independently with the same probability  $P$ . Then, the probability of failure can be calculated from the binomial distribution: Let  $M$  be the number of failed elements.  $M$  has binomial distribution with parameters  $n$  and  $P$ , hence its probability mass function is given by

$$P_{M;n}(m) = \binom{n}{m} P^m (1 - P)^{n-m} \quad (5)$$

where  $\binom{n}{m} = \frac{n!}{m!(n-m)!}$  is the binomial coefficient. The probability of failure of the system is

$$\begin{aligned}
P[\text{system failure}] &= P[M \geq m] \\
&= \sum_{i=m}^n P_{M;n}(i) \\
&= 1 - F_{M;n}(m-1)
\end{aligned} \tag{6}$$

Where  $F_{M;n}(m) = P[M \leq m]$  is the cumulative distribution function of  $M$ .

Example 1 Consider the case of a car with one spare tire. The car will become impaired if 2 (or more) tires are flat. In a conservative approximation, one may assume that all 5 tires are simultaneously used and subject to punctures. Then the probability of not completing a trip is given by Eqs. 5 and 6, with  $n = 5$ ,  $m = 2$ , and  $P$  = probability of puncture of a single tire during the trip.

**Problem 1.2** Compare the probability of completing a car trip in the cases without spare tire and with 1 spare tire by using Eq. 6 with  $(n = 4, m = 1)$  and  $(n = 5, m = 2)$ . Make the comparison for  $P = 0.001, 0.01, 0.1$ . Comment on the results.

Example 2 In order to fly, an airplane needs at least half of its engines to be functioning. Suppose that, during any given flight, engines fail independently, with probability  $P$ . Would you be safer in an airplane with 1, 2, 3 or 4 engines?

Under the condition of independent and equally likely failures, the number of non-functioning engines at the end of a generic flight,  $M$ , has binomial distribution with probability mass function in Eq. 5. The probability  $P_n$  that an airplane with  $n$  engines is unable to fly is therefore

$$\begin{aligned}
P_1 &= P_{M;n=1}(1) = \binom{1}{1} P^1 (1-P)^0 = P \\
P_2 &= P_{M;n=2}(2) = \binom{2}{2} P^2 (1-P)^0 = P^2 \\
P_3 &= P_{M;n=3}(2) + P_{M;n=3}(3) = \binom{3}{2} P^2 (1-P)^1 + \binom{3}{3} P^3 (1-P)^0 = 3P^2 - 2P^3 \\
P_4 &= P_{M;n=4}(3) + P_{M;n=4}(4) = \binom{4}{3} P^3 (1-P)^1 + \binom{4}{4} P^4 (1-P)^0 = 4P^3 - 3P^4
\end{aligned} \tag{7}$$

**Problem 1.3** Plot the probabilities  $P_1$ ,  $P_2$ ,  $P_3$ , and  $P_4$  in Eq. 7 as functions of  $P$ , for  $P$  in the range  $[10^{-4}, 10^{-1}]$ . Comment on the results.

The previous analysis rests on the assumption that airplane engines fail independently. This is a rather unrealistic assumption, because in many cases a single “common cause” may induce simultaneous or serial failure of several engines. A more sophisticated but still relatively simple model is as follows. Suppose that potentially damaging events occur at Poisson times during a flight, with mean rate  $\lambda$ . When one such event occurs, each engine of the airplane fails with probability  $p$ , independently of the other engines. Also, failure or survival of an engine in different potentially damaging events are assumed to be independent events. Notice that, in this case, engine failures are conditionally independent given a potentially damaging event. However, unconditionally (during a generic flight), engine failures are probabilistically dependent (failure of one engine makes it more probable that other engines also failed, because an engine failure indicates that at least one damaging event occurred during the flight). For example, engine failures may be caused by malfunctionings of the electrical system or by encounters with bird flocks. When any such event occurs, it is likely that several engines are damaged.

Before we analyze airplane reliability through this revised model, we recall the following property of Poisson processes. Consider a primary Poisson process  $\{t_i\}$  with rate parameter  $\lambda$ . A secondary process  $\{t'_i\}$  is obtained by “independently thinning”  $\{t_i\}$ . This

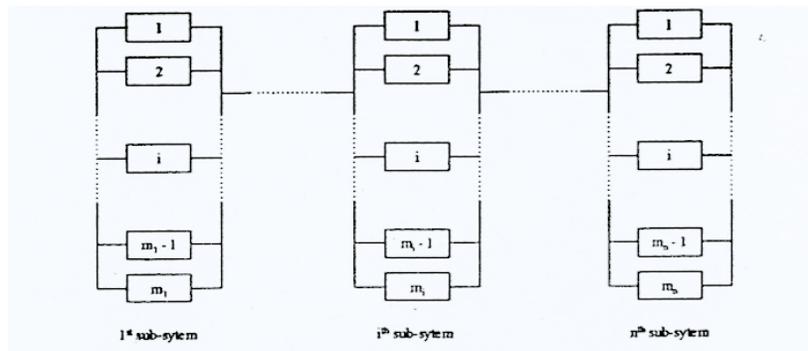
means that each point  $t_i$  is retained in  $\{t'_i\}$  with a given probability  $p$ , independently of all the other points. Then,  $\{t'_i\}$  is itself a Poisson process, with rate parameter  $\lambda' = \lambda p$ . Application of this result to our model shows that failure events of each given engine occur at Poisson times, with rate  $\lambda' = \lambda p$ . Considering a trip of duration  $T$ , the probability of failure of an individual engine is  $1 - e^{-\lambda p T}$ . This is also the probability of failure of a single-engine plane. Therefore, in order to make the results of the two models comparable, we should choose  $p$  in the present model so that  $1 - e^{-\lambda p T}$  equals  $P$  in the earlier model. This will make the reliability of a single engine plane during one trip the same in the two models.

To simplify the analysis of the present model, we make the realistic assumption that potentially damaging events are rare, so that, during a single flight, the probability of two or more such events is negligible relative to the probability of one event. This means that we exclude from the analysis events like “an airplane with two engines does not make the trip due to the occurrence of two potentially damaging events, each producing failure of one engine”. Also notice that, given a potentially damaging event, the probability of airplane failure is still given by the expressions in Eq. 7, with  $p$  in place of  $P$ . In order to obtain the probability of airplane failure in a flight of duration  $T$ , those probabilities must be multiplied by  $1 - e^{-\lambda T}$ , which is the probability of at least one potentially damaging event.

**Problem 1.4** Let  $T = 6$  hours and  $\lambda = 1/(10^4$  hours). Plot the probability of airplane failure against  $P = 1 - e^{-\lambda T}$  for  $P$  in the range  $[10^{-4}, 10^{-1}]$ , separately for a plane with 1, 2, 3, and 4 engines. Compare the results with those for independent engine failures (Problem 1.3).

### 3. MORE COMPLICATED SYSTEM CONFIGURATIONS

In even more complex cases, series and parallel connections are intermixed. For example, in assembling a car, it is necessary that a large number of components be simultaneously available (this is a series system and is highly vulnerable because the unavailability of just one component may force an assembly plant to shut down, as one knows well from labor strikes). To increase system reliability, car manufacturers rely on several alternative part suppliers and part-producing plants, which are used “in parallel”. This corresponds to a scheme with several subsets of components (sub-systems). The sub-systems are connected in series, but have an internal parallel structure, as illustrated by the following scheme.



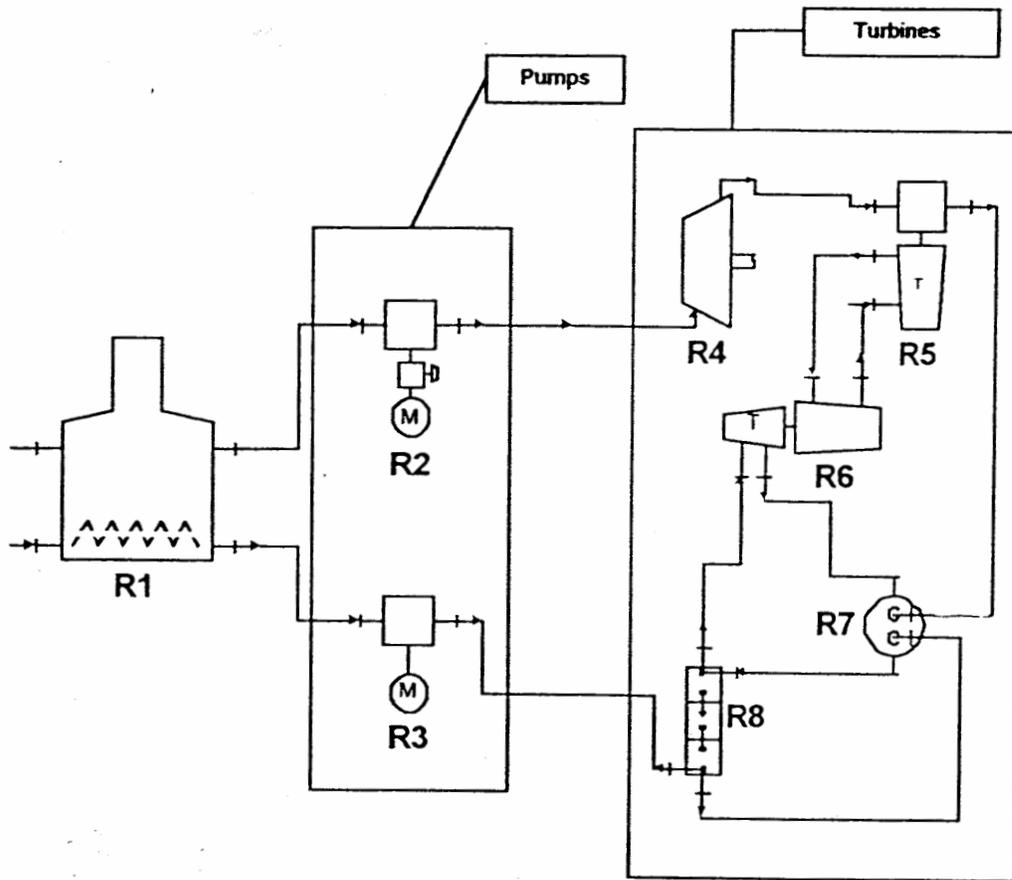
*Schematic illustration of a parallel-series system*

Suppose that there are  $n$  sub-systems and that sub-system  $i$  has  $m_i$  elements connected in parallel. Also denote by  $P_{ij}$  the probability of failure of the  $j^{\text{th}}$  element of sub-system  $i$ . In the simple case of independent element failures, the probability of failure of the system is

$$\begin{aligned}
 P[\text{system failure}] &= 1 - \prod_{i=1}^n P[i^{\text{th}} \text{ sub-system survives}] \\
 &= 1 - \prod_{i=1}^n \left( 1 - \prod_{j=1}^{m_i} P_{ij} \right)
 \end{aligned} \tag{8}$$

The following is an example of reliability analysis for a moderately complex system, which includes series and parallel connections, as well as m-out-of-n sub-systems.

Example Consider a system composed of a heater ( $R_1$ ), two pumps ( $R_2$  and  $R_3$ ), and 5 turbines ( $R_4$  through  $R_8$ ). The two pumps work in parallel, meaning that the pump sub-system operates if at least one of the pumps operates. The turbine sub-system operates if at least 3 turbines operate. The heater, pump, and turbine sub-systems are connected in series, meaning that they must all properly work for the whole system to perform adequately.



Between two scheduled maintenances, each component may fail independently of the others, with the following probabilities

R1	R2	R3	R4	R5	R6	R7	R8
0.05	0.10	0.08	0.20	0.17	0.09	0.15	0.15

The reliability of the system is defined as the probability that the system does not fail between scheduled maintenances. We denote by  $W_i$  the event “component  $i$  is working properly”. To calculate system reliability, we first consider the reliability of each sub-system separately:

- heater sub-system: Reliability =  $P[W_1] = 1 - 0.05 = 0.95$

- pump sub-system:

$$\text{Reliability} = 1 - P[W_2^c \cap W_3^c] = 1 - P[W_2^c]P[W_3^c] = 1 - (0.10)(0.08) = 0.992$$

- turbine sub-system: All possible events for which the turbine sub-system is in working conditions and their probabilities are listed below. Since the events are mutually exclusive, the probability of their union (which is the same as the event “the turbine sub-system works properly”) is the sum of their individual probabilities, i.e. 0.9733.

R4	R5	R6	R7	R8	P[W(turbines)]
Works	Works	Works	Works	Works	43.66%
Fails	Works	Works	Works	Works	10.91%
Works	Fails	Works	Works	Works	8.94%
Works	Works	Works	Works	Fails	7.70%
Works	Works	Works	Fails	Works	7.70%
Works	Works	Fails	Works	Works	4.32%
Fails	Fails	Works	Works	Works	2.24%
Fails	Works	Works	Works	Fails	1.93%
Fails	Works	Works	Fails	Works	1.93%
Works	Fails	Works	Works	Fails	1.58%
Works	Fails	Works	Fails	Works	1.58%
Works	Works	Works	Fails	Fails	1.36%
Fails	Works	Fails	Works	Works	1.08%
Works	Fails	Fails	Works	Works	0.88%
Works	Works	Fails	Works	Fails	0.76%
Works	Works	Fails	Fails	Works	0.76%
					<b>97.33%</b>

Finally, the reliability of the entire system is  $(0.95)(0.992)(0.9733) = 0.9172$ .

**Problem 1.5** Turbines are very expensive. Therefore, there are financial incentives to removing one of them. How would removal of turbine R4, which is the least reliable one, affect the reliability of the whole system?

#### 4. SYSTEMS THAT “SHARE THE LOAD”

Many physical and non-physical systems work “in parallel” in a sense different from the case considered above. Rather than just having multiple elements connected in parallel and requiring that at least one works, the elements share the total applied “load” or demand. Parallel systems of this type may function in different ways, depending on the characteristics of the components. The following are examples of two different types:

- A first example is a power company that interchangeably uses several power generating plants to meet total demand. In this case, the system fails if the demand exceeds the combined capacity of the power plants; i.e., the system capacity is the sum of the capacities of its components;
- A second example is a rope made of several bundles that share the applied load. If the bundles are “ductile” (e.g., they are made of mild steel), they are able to redistribute the load among themselves. In this case the strength of the rope is the sum of the bundle strengths and the behavior of the system is of the same type as that of the power company in the previous example. However, if the bundles are brittle (e.g. they are made of glass), then each bundle will break as soon as its capacity is reached. The load must then be carried by the surviving bundles. In this case, the strength of the rope is less than the sum of the strengths of the individual components.

For a system of the first type, the probability of failure is

$$P[\text{system failure}] = P[D > C_1 + C_2 + \dots + C_n] \quad (9)$$

where  $D$  is the demand and  $C_1, \dots, C_n$  are the capacities of the components. We shall see later in the course how to evaluate probabilities of the type in Eq. 9, when  $D$  and the  $C_i$  are random variables.

Systems of the second type are more complicated to analyze, because the capacity of the system is a complicated nonlinear function of the component capacities. A way to evaluate reliability in this and other complicated cases is to use Monte Carlo simulation.