

1.050: Energy Bounds in Elasticity (HW#10)

Due: Friday, November 30, 2007

MIT – 1.050 (Engineering Mechanics I)

Fall 2007

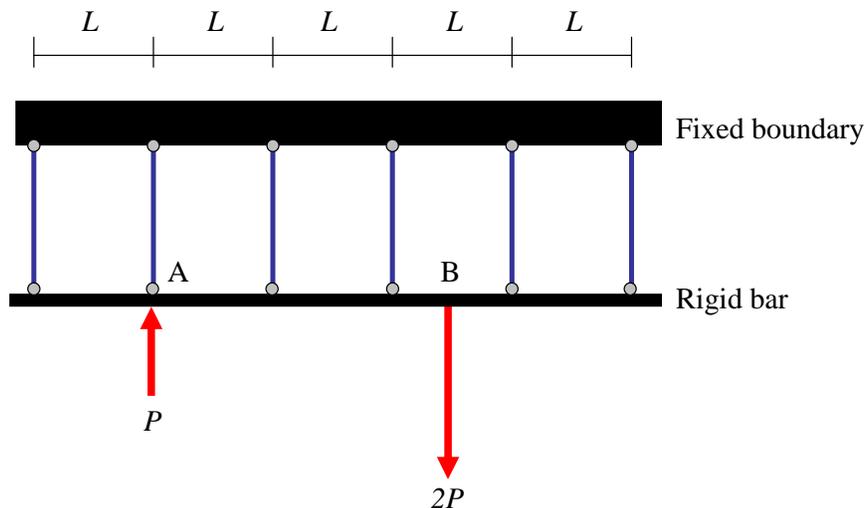
Instructor: Markus J. BUEHLER

Team Building and Team Work: We strongly encourage you to form Homework teams of three students. Each team only submits one solution for correction. We expect true team work, i.e. one where everybody contributes equally to the result. This is testified by the team members signing at the end of the team copy a written declaration that "the undersigned have equally contributed to the homework". Ideally, each student will work first individually through the homework set. The team then meets and discusses questions, difficulties and solutions, and eventually, meets with the TA or the instructor. Important: Specify all resources you use for your solution.

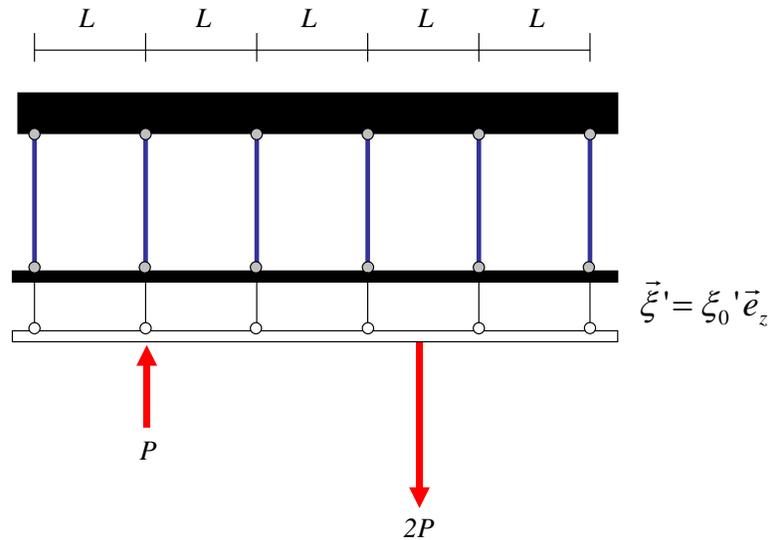
The following set of exercises is designed to familiarize you with the use of energy bounds in linear elasticity. This problem set is focused on 1D methods, and you will be able to apply the techniques you have learned in Lectures 27, 28 and 29. Please review the lecture notes and handouts carefully.

For some of the solutions it is recommended to use *Maple* or *Mathematica* software to solve the resulting system of linear equations.

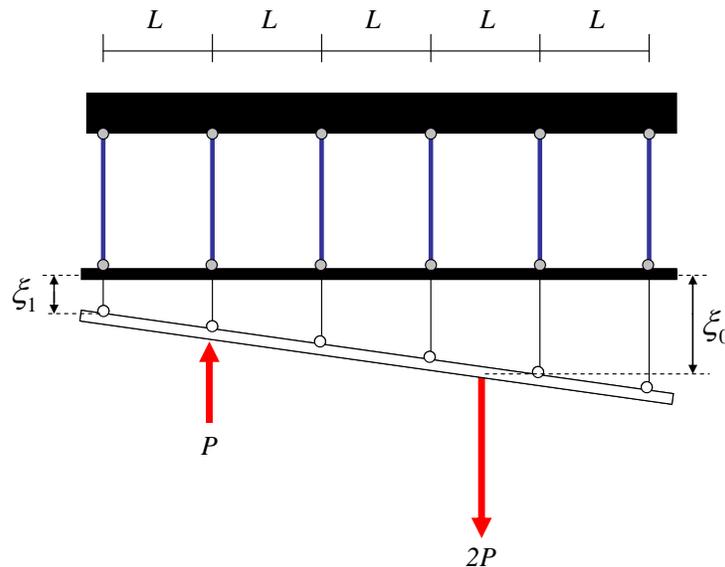
- 1. Truss structure:** The sketch below shows a truss structure constructed with 6 vertical members and a horizontal rigid bar. The structure is subjected to two loads at points A and B (thus, this is a *force-driven* experiment). The top of the truss structure is connected to a fixed boundary. The goal of this problem is to solve for all displacements and axial forces using the structure's energy content.



- a. *Proposed, approximate solution 1:* Use the displacement field $\vec{\xi}' = \xi_0' \vec{e}_z$, which is displayed in the following sketch:



- i. Confirm that this displacement field is kinematically admissible.
 - ii. Compute the potential energy for the truss system, given this displacement field.
 - iii. Use the theorem of minimum potential energy to determine ξ_0' and the minimum potential energy, $\min_{\xi_0'} E_{pot}(\xi_0')$.
 - iv. Calculate the forces in each truss member. Check if the resulting force field is statically admissible (**Hint:** carry out a moment and force balance). Discuss the results.
- b. *Proposed solution 2:* Derive the displacement field which matches the kinematic conditions displayed in the following sketch:

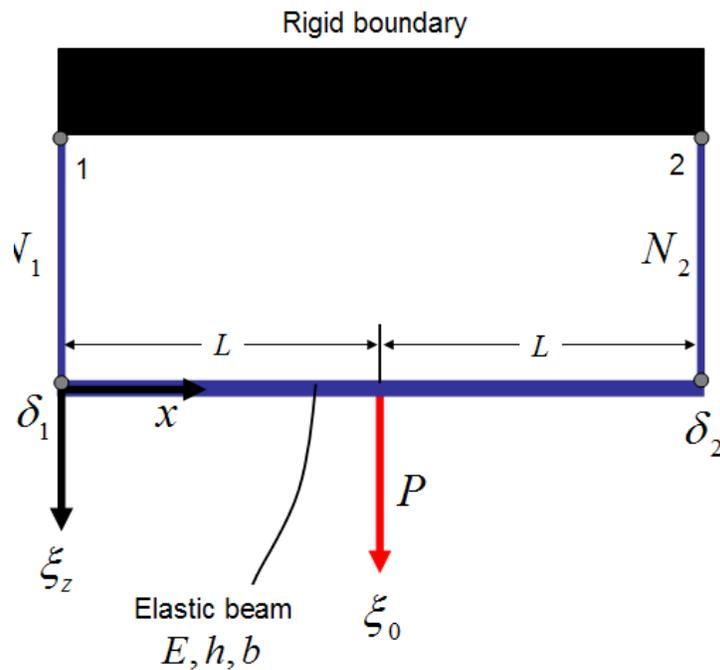


- i. Compute the potential energy for the truss system, given this displacement field.
 - ii. Use the theorem of minimum potential energy to determine ξ_0, ξ_1 and the minimum potential energy, $\min_{\xi_0} E_{pot}(\xi_0)$.
 - iii. How does the solution for the potential energy for ξ_0 compare with the one obtained in Part a? Discuss your finding.
 - iv. Calculate the forces in each truss member. Check if the resulting force field is statically admissible. Discuss the results.
- c. For the same force-driven experiment of Parts a and b,
- i. Compute the complementary energy for the truss system.
 - ii. Use the theorem of minimum complementary energy to determine the axial forces and the minimum complementary energy, $\min_{N_i} E_{com}(N_i)$.
 - iii. Compare the solution of Part c,ii to the solution of Part b,ii. What do we conclude about the quantities ξ_0, ξ_1, N_i ? Is the solution from Part a for ξ_0 ' an upper or a lower bound to the actual solution ξ_0 ?

2. **Minimum potential energy approach:** The photograph below shows a section of a suspension bridge. We will use what we have learned about truss structures, beam deformation, and energy methods to analyze the deformation behavior of the top cord and the vertical cables. The engineering model of a simplified structure is presented in the next sketch.

Photograph removed due to copyright restrictions.

Photograph adapted from: O'Donnell, P.S. (2007) Akashi Kaikyo Suspension Bridge, in David Denenberg, www.bridgemeister.com, <http://www.bridgemeister.com/pic.php?pid=996>.



Note: Please take a look at the [handout of Lecture 29](#), which summarizes the approach for another example. Note that the total length of the elastic beam is $2L$.

Solve this problem using the potential energy approach. Assume that the displacement of the elastic beam is of the form:

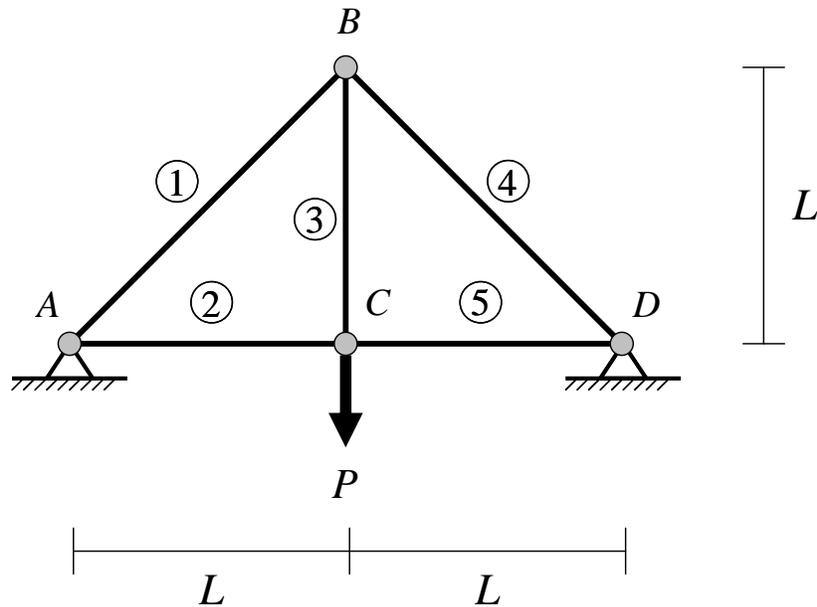
$$\xi_z(x) = \alpha + \beta \left(\frac{x-L}{L} \right)^2$$

- a. Write the free energy of the system. To do this,
 - i. First write the displacements of the three points: 1) at member 1, 2) at member 2, and 3) where the load is applied, as a function of the displacement function parameters α, β .
 - ii. Express the free energy of the two trusses as a function of the parameters in the displacement function.
 - iii. Express the free energy of the elastic beam as a function of the parameters in the displacement function. Start from the free energy expression of an elastic beam:

$$\psi_B = \int_{x=0}^{2L} \int_{z=-h/2}^{h/2} \int_{y=-b/2}^{b/2} \frac{1}{2} E (\varepsilon_{xx}^0 + \mathcal{G}_y^0 z)^2 dy dz dx \quad (\text{see handout for Lecture 29})$$

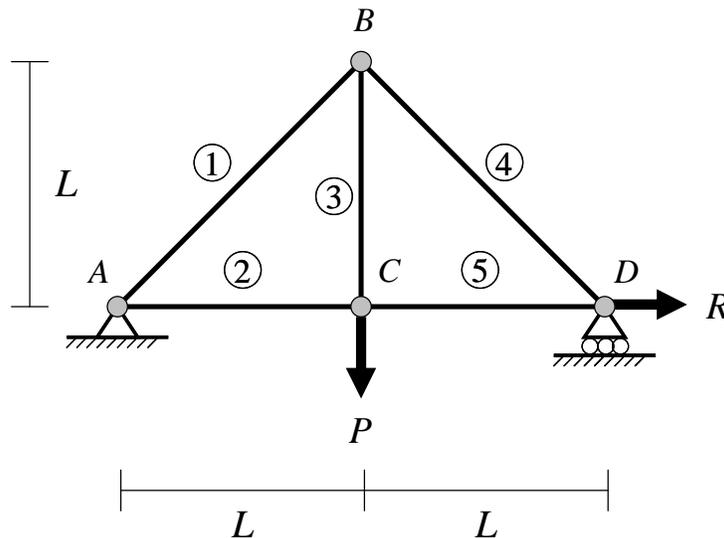
- iv. Express the external work as a function of the parameters in the displacement function.
 - v. Write the expression for the potential energy $E_{\text{pot}}(\alpha, \beta)$.
- b. Using the minimum potential energy theorem, find the solution for α, β that approximates the structure deformation solution. Calculate the potential energy for this set of parameters. Is the potential energy an upper or lower bound of the absolute minimum of the potential energy?
 - c. Plot the displacement function, for $E = 200$ GPa, $h = 0.05$ m, $b = 0.05$ m, $L = 1$ m (h, b, L are dimensions of the beam element), $S = b^2$, $H = 2$ m (S, H are dimensions of the truss elements), $P = 100$ kN, using Matlab, Excel, Maple, or Mathematica. Do not hand in a hand-drawn sketch. Also provide the maximum displacement at the point where the force is applied.
 - d. Is the displacement ξ_0 an upper bound or lower bound to the actual solution? (Do not calculate the actual solution; simply discuss this point).
 - e. Calculate the forces in each truss member, and check if the solution is statically admissible. Discuss the results.
 - f. In a few sentences, discuss a possible strategy to arrive at the exact solution to this problem.

3. **Solving a statically indeterminate truss:** The objective of this exercise is to determine all the member forces for the truss structure shown below. The truss is loaded with a force P at node C . All members in the truss structure have the same cross-section parameters (E , Young's modulus and S , the cross-section area). For the solution of this exercise, use the given notation for nodes (A through D) and truss members (1 through 5).



We take advantage of the great tools of energy bounds in elasticity to solve this statically indeterminate truss by applying the *complementary energy approach* to the force-driven problem (recall part 1.3 in Lecture 29, see also slide 11 in the handout for Lecture 29).

- Determine the degree of indeterminacy for the truss structure.
- Solve for the member forces of the following alternative truss structure:



- c. Compute the complementary energy:

$$E_{com}(F_j, \vec{R}) = \Psi^*(F_j) - \bar{\xi}^d \cdot \vec{R}$$

Choose carefully your boundary conditions in order to replicate the original structure.

- d. Use the theorem of minimum complementary energy $\min_{F_j} E_{com}(F_j)$ to determine all member forces.