

# PROBLEM SET 2 - SOLUTIONS

## Comments about Problem Set 2

### PROBLEM 1:

- Please keep in mind what the meaning of “line of action of a force” is. When you calculate the resultant pressure force, you impose it to cause the same total force (1), and to cause the same moment around any point (2) as the real pressure distribution. Condition (1) yields the magnitude of the resultant force; condition (2) yields the line of action. The  $h/3$  (or  $2h/3$ ) that we always use for the position of the resultant force equivalent to a triangular distribution comes from applying condition (2). A condition similar to (2) also applies to determine centers of gravity. So please remember condition (2) (i.e., equivalence of moments) whenever you have to determine lines of action of resultant forces or centers of gravity.

- Many groups forgot the width of the dam (100 m) in their calculations, and treat the dam as if it had unit width. A good way to realize about this kind of mistake is to check dimensions. If you forget the width, your force will have units of N/m instead of N. Note that even if you forget the width your value for “h” will be right, because the width appears in all terms and cancels out. (This is why we often study problems “per unit width”).

- Part b was graded considering whether your answer was consistent to your numerical result for the maximum admissible water depth, “h(max)”:

- If you got  $h(\text{max}) < 30$  m, you conclude that the dam was not properly designed, because it would topple over for high values of “h” in the dam.
- If you got  $h(\text{max}) > 30$  m, then you conclude that the dam is properly designed (and you need to explain why).
- If you got  $h(\text{max}) \gg 30$  m, then you conclude that the dam is not properly designed, because even if it is safe (it won’t topple over), it’s probably overdimensioned (i.e., you could build a cheaper dam with less material by reducing the base).

### PROBLEM 2:

- Please remember that pressure is a scalar magnitude, not a vector! With this I mean that, at a certain point, it has the same magnitude in all directions (it is isotropic). It always acts perpendicularly to surfaces (this is a consequence of the definition of pressure, and therefore it is not an assumption, as some groups wrote). Now, if pressure is not a vector, why do we draw all those little arrows to represent its direction? Those little arrows represent the direction of the pressure force on the surface, which is normal to it and therefore has a direction. The pressure itself is isotropic, so if we change the direction of the surface, the pressure force will change direction to become perpendicular to the surface, but its magnitude will remain constant. If you don’t have these concepts clear, you should review the notes from Recitation 1 and Lectures 3 and 4.

### PROBLEM 3:

- Please avoid writing your results with too many decimal values. Remember that your results have a precision, and are not totally exact (e.g., you know gravity with an accuracy of  $\pm 0.5\%$ , and most of your data have limited precision as well). So, for instance, don’t write that the raft is submerged to a depth of 0.496585365 m, but to 0.497 m. The second value makes more sense, because you are acknowledging that you don’t have enough precision to determine the four decimal place.

- Some groups didn't know the meaning of the metacenter. They confused it with the center of buoyancy or the center of gravity. Please refer to the typed notes from Recitation 2 and to the solution to the problem a few pages below.

- A useful thing to remember from this problem is the following shortcut to calculate moments of inertia in symmetric sections: If the section is symmetric with respect to "x" or "y" (just one is enough), then it is always true that  $I_{xy} = 0$  (since  $I_{xy}$  involves odd integration both in "x" and in "y"). Besides, it is also always true that  $I_{xx}$  and  $I_{yy}$  are the principal moments of inertia, meaning that one is the maximum and the other is the minimum. (This property is usually taught in Freshman Physics). In this particular case, where  $I_{xx} = I_{yy}$ , the maximum and the minimum are the same, so the moment of inertia with respect to any other axis is also the same.

#### **Problem #4**

Generally, everyone did well with the derivations. The conclusions reached about the accuracy of the approximation (U) in part c could be made stronger by providing some error calculations. With the exception of one or two groups, all should make neater graphs (use Excel if possible).

#### **Problem #5**

Generally, everyone did well on this problem. A few groups got tripped up on setting up the manometry and Bernoulli equations. For manometry, it's a good idea to start with pressure at one point and work your way around to pressure at another point so that your signs are correct and to help ensure that you don't leave anything out. It's easiest to use

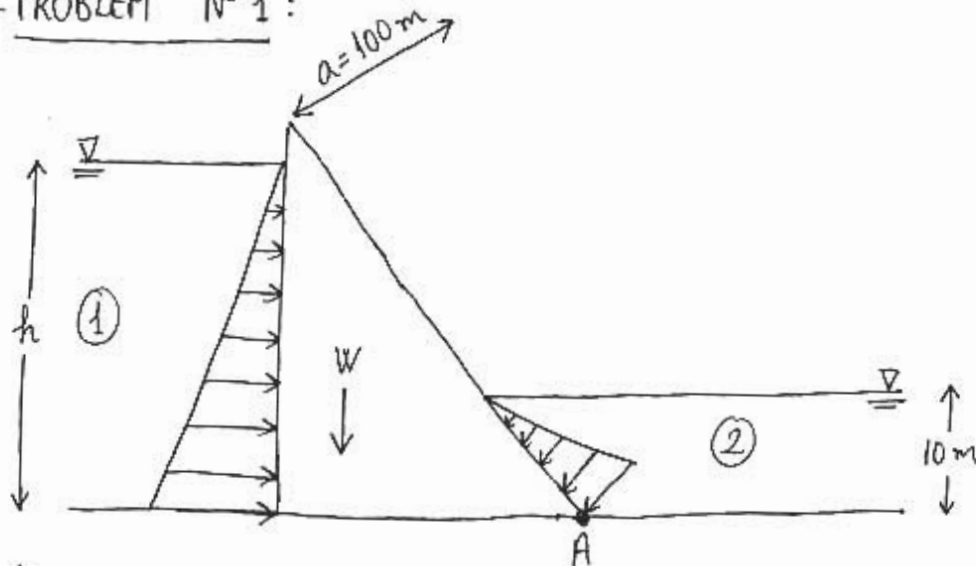
Bernoulli along the streamline that runs through the centerline of the pipe; in this case, you can cancel out the  $(\rho \cdot g \cdot z)$  terms on both sides since  $z_a = z_b$ . One other thing to keep in mind to make future calculations easier is that the density of air can usually be neglected in a problem like this because the density of air is much less than the density of water. (Note that I did not take points off if you included the density of air in your calculations). It's also always a good idea to check the units of your answer to make sure that your answer makes sense.

#### **Problem #6**

The concept that gave most groups trouble with this problem was in part b. Without any flow, the pressure would vary hydrostatically and we could use Bernoulli to find that  $p_b$  is about 3780 Pa greater than  $p_a$ . If the flow were to be directed uphill, the pressure at b would need to exceed this amount to overcome the effects of gravity. Since the pressure difference between a and b is negligibly small (compare pressure difference to atmospheric pressure), the flow will be directed from a to b (downhill). Some groups also did not fully explain why the pressure difference between A and B would be the same as between A' and B'. The flow is well-behaved, and the distances between A - A' and B - B' is equal. You can thus show using Bernoulli that pressure difference between A and B is the same as between A' and B'.

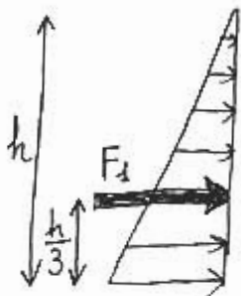
# PROBLEM SET 2 - SOLUTIONS

- PROBLEM N° 1:



a) The dam will topple over about A when the total moment about A is in clockwise direction.

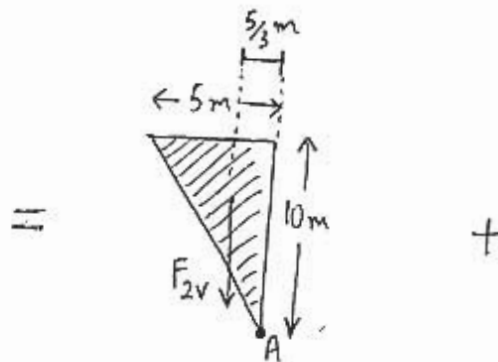
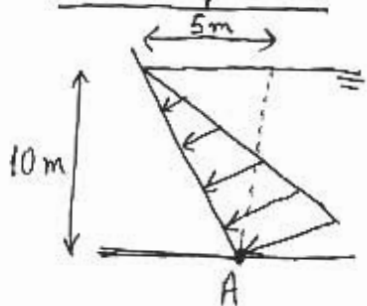
- Pressure force (1)



$$F_1 = \frac{1}{2} \rho g h \cdot h \cdot a = \frac{\rho g h^2 a}{2}$$

$$(\rho = \rho_{\text{water}} = 1000 \frac{\text{kg}}{\text{m}^3})$$

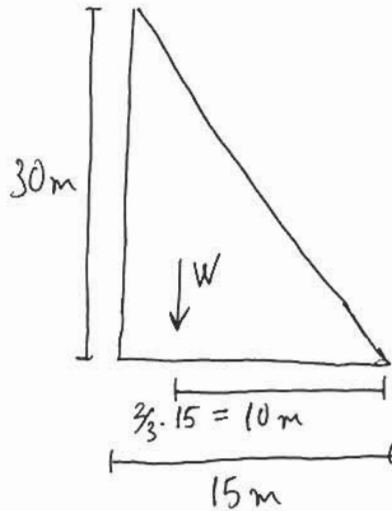
- Pressure force (2)



$$F_{2v} = \text{Weight of water above} = \rho g \frac{1}{2} \cdot 5 \cdot 10 \cdot a = 25 \rho g a$$

$$F_{2H} = \frac{1}{2} \rho g 10 \cdot 10 \cdot a = 50 \rho g a$$

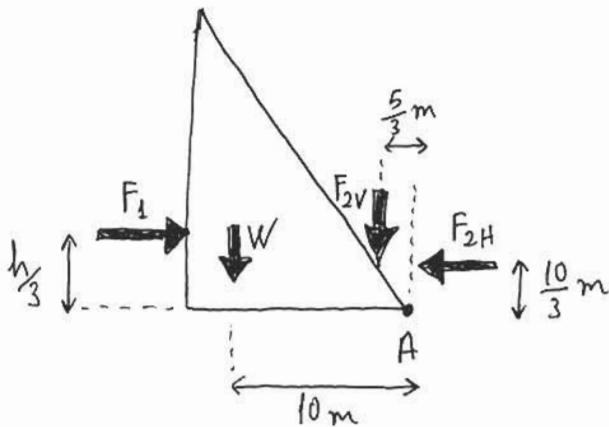
- Weight of the dam:



$$W = \rho_c g \frac{1}{2} 30 \cdot 15 \cdot a = 225 \rho_c g a$$

$$\rho_c = 2300 \frac{\text{kg}}{\text{m}^3}$$

MOMENTS ABOUT A:



$$\begin{aligned} \overset{\curvearrowright}{M}_A &= -\frac{h}{3} F_1 + 10 W + \frac{5}{3} F_{2v} + \frac{10}{3} F_{2H} = \\ &= g a (-166'67 h^3 + 5'1750 \cdot 10^6 + 4'167 \cdot 10^4 + \\ &\quad + 1'667 \cdot 10^5) = \\ &= g a (-166'67 h^3 + 5'3834 \cdot 10^6) \end{aligned}$$

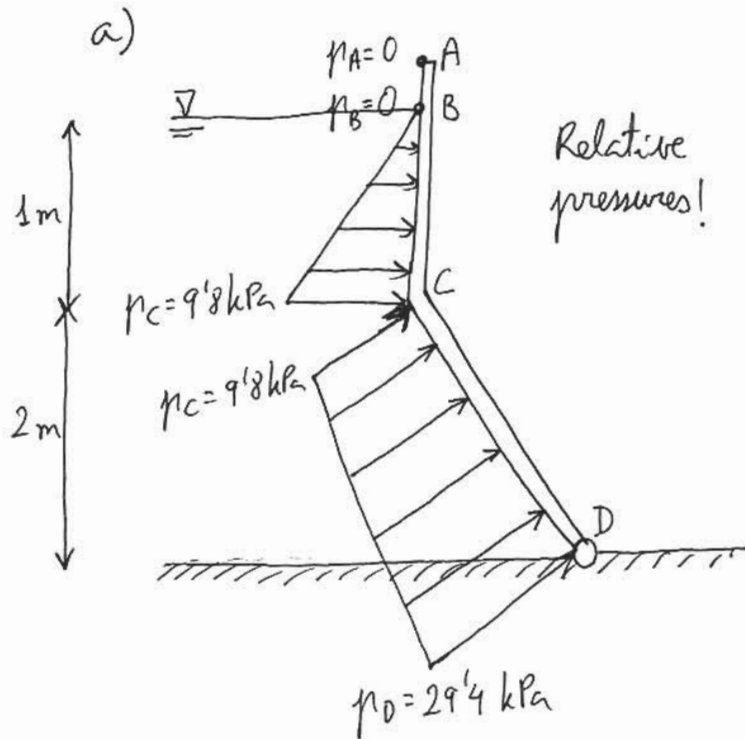
$$\overset{\curvearrowright}{M}_A \geq 0 \Rightarrow -166'67 h^3 + 5'3834 \cdot 10^6 \geq 0 \Rightarrow \underline{\underline{h \leq 31'85 \text{ m}}}$$

The dam will topple over about A for  $h \approx 31'8 \text{ m}$

b)

The dam is properly designed. The depth of water ( $h$ ) necessary to topple the dam exceeds the height of the dam (30 m). This means the dam cannot topple over. For  $h > 30 \text{ m}$ , the water will spill over the dam and increase the depth on the backside, which helps stabilize the dam.

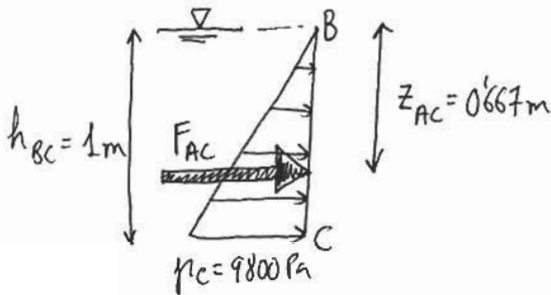
- PROBLEM N° 2:



$$\begin{aligned} \underline{p_A} &= p_{\text{ATM, gauge}} = \underline{0} \\ \underline{p_B} &= p_{\text{ATM, gauge}} = \underline{0} \\ \underline{p_C} &= p_B + \rho g h_{BC} \\ &= 0 + 9800 \cdot 1 = \underline{9800 \text{ Pa}} \\ \underline{p_D} &= p_B + \rho g h_{BD} \\ &= 9800 \cdot 3 = \underline{29400 \text{ Pa}} \end{aligned}$$

b)

The total force on AC is the volume of the pressure prism, i.e., (area of the pressure triangle)  $\times$  (length of the gate)



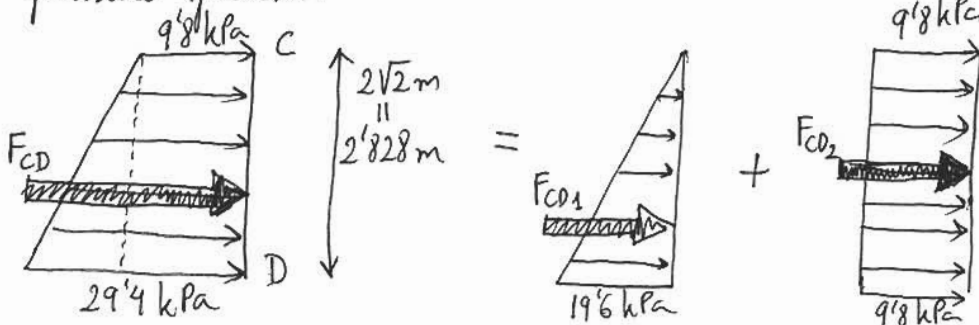
$$\underline{F_{AC}} = \left( \frac{1}{2} \cdot 9800 \cdot 1 \right) \cdot 1 = \underline{4900 \text{ N}}$$

It acts on the CG of the triangle,

$$\underline{z_{AC} = \frac{2}{3} h_{BC}}, \quad \underline{\text{towards the right}}$$

c)

Again, the total force on CD is the volume of the pressure prism:

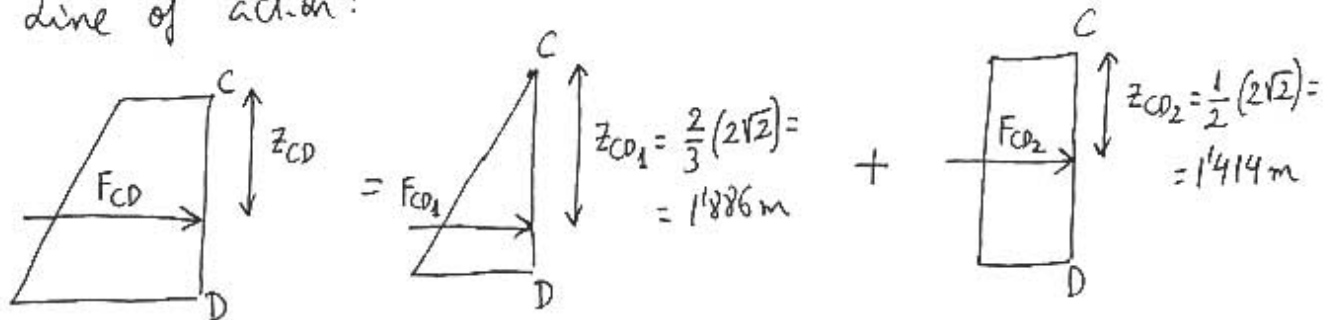


$$F_{CD_1} = \frac{1}{2} \cdot 19600 \cdot 2\sqrt{2} \cdot 1 = 27719 \text{ N} \quad \triangleleft$$

$$F_{CD_2} = 9800 \cdot 2\sqrt{2} \cdot 1 = 27719 \text{ N} \quad \boxtimes$$

$$\underline{F_{CD}} = F_{CD_1} + F_{CD_2} = \underline{\underline{55438 \text{ N}}} \quad \boxtimes$$

Line of action:

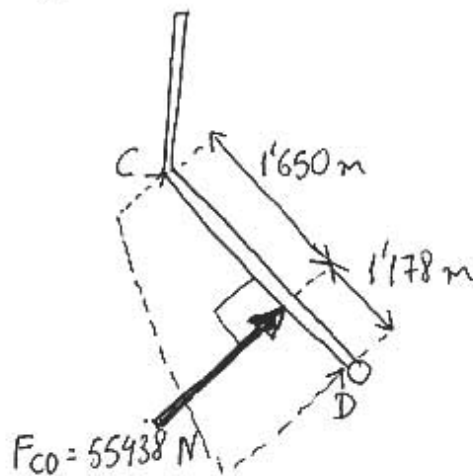


Taking moments with respect to C:

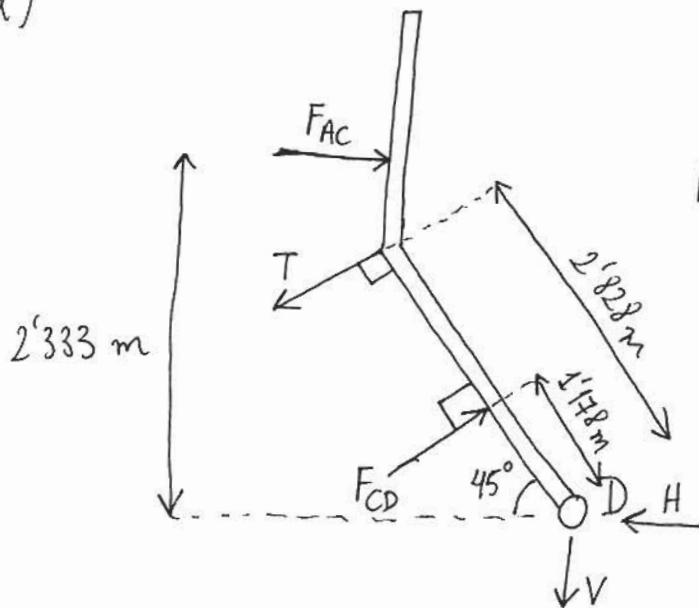
$$\overset{\curvearrowleft}{M}_C = F_{CD} \cdot z_{CD} = F_{CD_1} \cdot z_{CD_1} + F_{CD_2} \cdot z_{CD_2} \Rightarrow$$

$$\Rightarrow \underline{\underline{z_{CD}}} = \frac{27719 \cdot 1'886 + 27719 \cdot 1'414}{55438} = \underline{\underline{1'650 \text{ m}}}$$

Graphically:



d)



(positive for tension)  
 T: Force on the wire  
 H, V: Forces exerted by the hinge on the gate.

Since the gate is hinged at D, the total moment about D must be zero:

$$\begin{aligned} \overset{\curvearrowright}{M}_D &= -F_{AC} \cdot 2'333 + T \cdot 2'828 - F_{CD} \cdot 1'178 = 0 \Rightarrow \\ \Rightarrow T &= \frac{4900 \cdot 2'333 + 55438 \cdot 1'178}{2'828} = \underline{\underline{27135 \text{ N}}} \end{aligned}$$

e)

Equilibrium of horizontal forces:

$$\begin{aligned} F_{AC} - T \cos 45^\circ + F_{CD} \cos 45^\circ - H &= 0 \Rightarrow \\ \Rightarrow H &= 4900 - 27176 \frac{\sqrt{2}}{2} + 55438 \frac{\sqrt{2}}{2} = \underline{\underline{24884 \text{ N}}} \end{aligned}$$

Horizontal force exerted by the hinge on the gate.

→ The horizontal force on the hinge is of the same magnitude, acting towards the right.

Equilibrium of vertical forces

$$-T \sin 45^\circ + F_{CD} \sin 45^\circ - V = 0 \Rightarrow V = -27176 \frac{\sqrt{2}}{2} + 55438 \frac{\sqrt{2}}{2} = \underline{\underline{19984 \text{ N}}}$$

→ The vertical force on the hinge has the same magnitude but acts upwards.

Vertical force exerted by the hinge on the gate.

- PROBLEM N°3:

a) It was rather clever, indeed. The raft is designed to maximize the moment of inertia (thus yielding a large metacentric height and improving stability). The moment of inertia with respect to, say, the x-axis is given in 2-D by

$$I_{xx} = \iint_A y^2 dA$$

Since the surface area of the beams is fixed, we maximize  $I_{xx}$  by maximizing "y", i.e., by placing the beams as far from the center of gravity of the section as possible.

b) The total mass of the raft is:

$$M_{\text{raft}} = \underbrace{4 \cdot 0.5 \cdot 0.5 \cdot 5 \cdot 800}_{= 4000 \text{ kg}} + 100 + 300 + 100 + 100 =$$

wooden beams                      mat    shed    mat    food

$$= 4600 \text{ kg}$$

When the wooden beams are totally submerged, the buoyancy force is:

$$F_{B, \text{max}} = \rho_{\text{seawater}} \cdot g \cdot (\text{Volume of the beams}) =$$
$$= 1025 \cdot 9.8 \cdot (4 \cdot 0.5 \cdot 0.5 \cdot 5) = 50225 \text{ N}$$



For the raft to float, the total weight (raft + students) has to be less than or equal to  $F_{B, \max}$ , i.e.,

$$W \leq F_{B, \max}$$

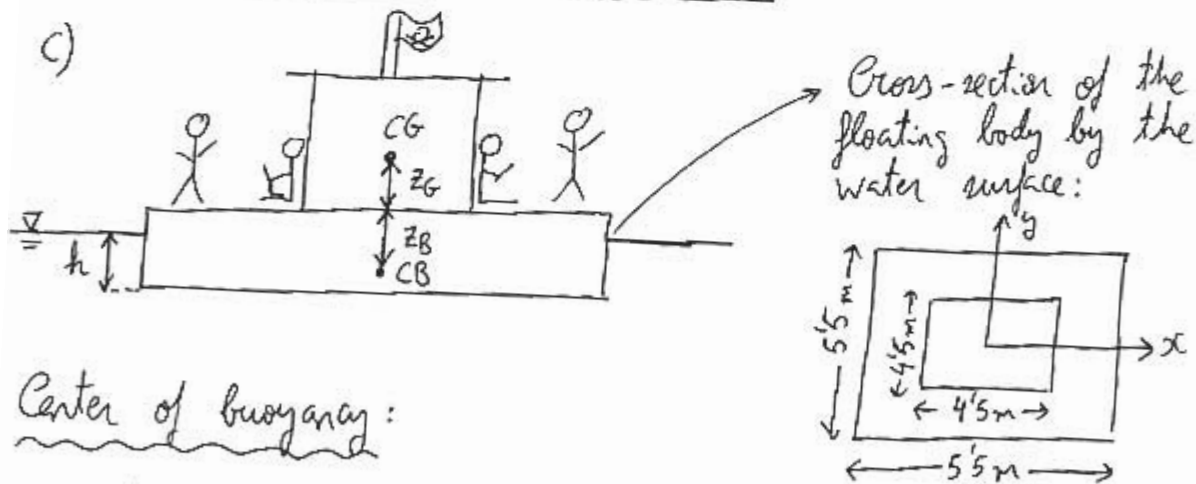
$$(M_{\text{raft}} + m_{\text{students}}) \cdot g \leq F_{B, \max}$$

$$(4600 + m_{\text{students}}) \cdot 9.8 \leq 50225$$

$$m_{\text{students}} \leq \frac{50225}{9.8} - 4600 = 525 \text{ kg}$$

$$\# \text{ students} \leq \frac{525 \text{ kg}}{70 \text{ kg/student}} = 7.5 \text{ students} \Rightarrow$$

$\Rightarrow 7$  students can get on board



Center of buoyancy:

$$\begin{aligned} \text{Total weight} \rightarrow W &= (4600 + 7 \cdot 70) \cdot 9.8 = \\ &= 49882 \text{ N} = F_B \text{ (Buoyancy force)} \end{aligned}$$

$$F_B = (\text{Submerged volume}) \cdot \rho_{\text{seawater}} \cdot g$$

$$49882 = (4 \cdot 0.5 \cdot h \cdot 5) \cdot 1025 \cdot 9.8 \Rightarrow h = 0.497 \text{ m} (< 0.5 \text{ m } \checkmark)$$

$$z_B = (0.5 - h) + \frac{h}{2} = 0.252 \text{ m}$$

Center of gravity:

$$z_G = \frac{\sum m_i \cdot z_i}{\sum m_i} = \frac{4000 \cdot (-0.25) + 100 \cdot 0 + 300 \cdot 1 + 100 \cdot 2 + 100 \cdot 0.2 + 7.70 \cdot 0.8}{4600 + 7.70} = -0.017 \text{ m}$$

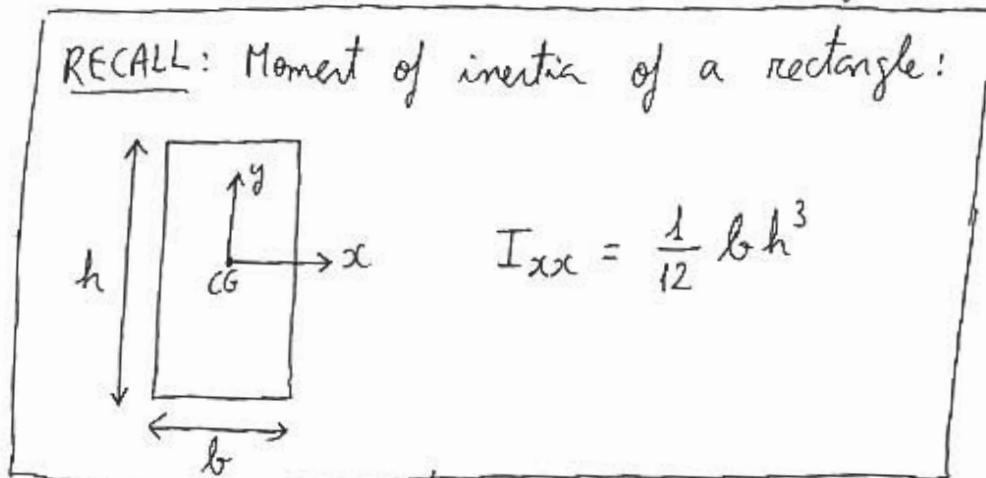
Metacenter

As explained in the notes from RECITATION 2, the height of the metacenter above the center of buoyancy is

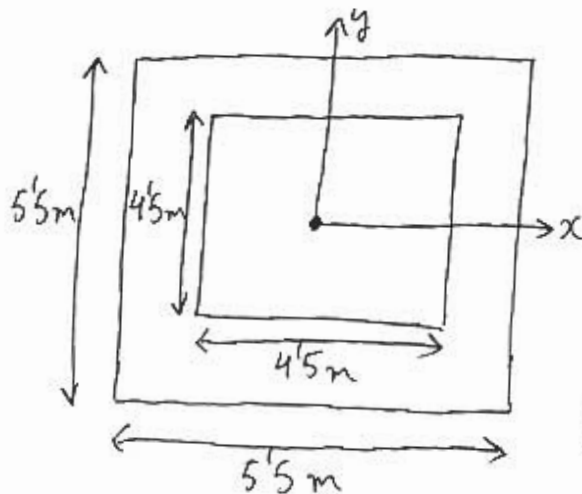
$$h_{CB \rightarrow M} = \frac{I_{xx}}{V_D}$$

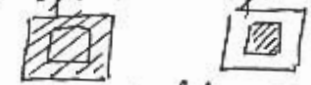
$I_{xx}$ : Moment of inertia of the cross-section of the floating body at the water level

$V_D$ : Submerged volume



For our cross-section:

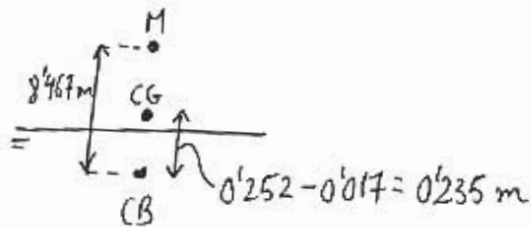


$$I_{xx} = \underbrace{\frac{1}{12} 5.5^4}_{I_{xx} \text{ exterior square}} - \underbrace{\frac{1}{12} 4.5^4}_{I_{xx} \text{ interior square}} = 42.08 \text{ m}^4$$


(You can also calculate  $I_{xx} = \iint_A y^2 dA$  by integration, but it takes much longer!!)

$$V_D = 4 \cdot 0.5 \cdot \underbrace{0.497}_h \cdot 5 = 4.97 \text{ m}^3$$

$$h_{CB \rightarrow M} = \frac{42.08}{4.97} = 8.467 \text{ m}$$



height of M - height of CG  $\approx 8.2 \text{ m} \gg 0 \rightarrow$  STABLE

The metacenter is far above the CG and thus flotation is stable.

d)

Since the section is symmetric with respect to the x-axis,  $I_{xy} = 0$ , because:

$$I_{xy} = - \iint_{\text{area}} xy \, dA = - \left[ \iint_{\text{area } (y < 0)} xy \, dA + \iint_{\text{area } (y > 0)} xy \, dA \right] = 0$$

(The section is also symmetric w.r.t. the y-axis, and that alone would also imply  $I_{xy} = 0$ ).

From the geometry of the raft we see that  $I_{yy} = I_{xx}$ .

Therefore:

$$I_{\theta\theta} = I_{xx} \cos^2 \theta + I_{xx} \sin^2 \theta + 0 = I_{xx}$$

I.e., the moment of inertia w.r.t. any axis is equal to  $I_{xx}$ . All axes are equally unfavorable. Therefore, since we proved in part (c) that the raft is stable for disturbances about the x-axis, it is stable for any disturbance.

- PROBLEM N<sup>o</sup> 4:

a) Plugging (1) and (3) into (2)

$$\underbrace{z_0 \left(1 - \frac{y}{h}\right)}_z = \underbrace{\rho \kappa u_* (y+y_0) \left(1 - \frac{y}{h}\right)}_{V_x} \frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{z_0}{\rho \kappa u_*} \frac{1}{(y+y_0)}$$

Since  $u_* = \sqrt{z_0/\rho}$ ,  $\frac{z_0}{\rho \kappa u_*} = \frac{u_*}{\kappa}$

$$\boxed{\frac{\partial u}{\partial y} = \frac{u_*}{\kappa} \frac{1}{(y+y_0)}} \quad \text{Governing equation}$$

$$\int_0^y \frac{du}{dy'} dy' = \int_0^y \frac{u_*}{\kappa} \frac{1}{(y'+y_0)} dy'$$

$$u(y) - u(y=0) = \frac{u_*}{\kappa} \left[ \ln(y'+y_0) \right]_0^y$$

0 (NO-slip BC)

$$\boxed{u = \frac{u_*}{\kappa} \ln \frac{y+y_0}{y_0}} \quad 0 \leq y \leq h$$

b)

$$U = \frac{1}{h} \int_0^h u(y) dy = \frac{1}{h} \frac{u_*}{\kappa} \int_0^h \ln \frac{y+y_0}{y_0} dy$$

ASIDE: Computation of  $\int \ln \frac{y+y_0}{y_0}$  using integration by parts:

$$\int u dv = uv - \int v du; \text{ Take } u = \ln \frac{y+y_0}{y_0}, \quad dv = dy$$
$$\int \ln \frac{y+y_0}{y_0} dy = y \ln \frac{y+y_0}{y_0} - \int \frac{y}{y+y_0} dy = y \ln \frac{y+y_0}{y_0} - \int \left(1 - \frac{y_0}{y+y_0}\right) dy =$$
$$= y \ln \frac{y+y_0}{y_0} - y + y_0 \ln(y+y_0) + C$$

Therefore

$$\begin{aligned} U &= \frac{u_*}{\kappa h} \left[ y \ln \frac{y+y_0}{y_0} - y + y_0 \ln (y+y_0) \right]_0^h = \\ &= \frac{u_*}{\kappa} \left( \ln \frac{h+y_0}{y_0} - 1 + \frac{y_0}{h} \ln \frac{h+y_0}{y_0} \right) = \\ &= \frac{u_*}{\kappa} \left[ \left(1 + \frac{y_0}{h}\right) \ln \frac{h+y_0}{y_0} - 1 \right] \end{aligned}$$

For  $y_0 \ll h \Rightarrow \left(1 + \frac{y_0}{h}\right) \approx 1$  and  $\frac{h+y_0}{y_0} \approx \frac{h}{y_0}$ ,

$$U \approx \underbrace{\frac{u_*}{\kappa} \ln \frac{h}{y_0}}_{u_s = u(y=h)} - \frac{u_*}{\kappa} = \frac{u_*}{\kappa} \ln \frac{h}{ey_0}$$

c)

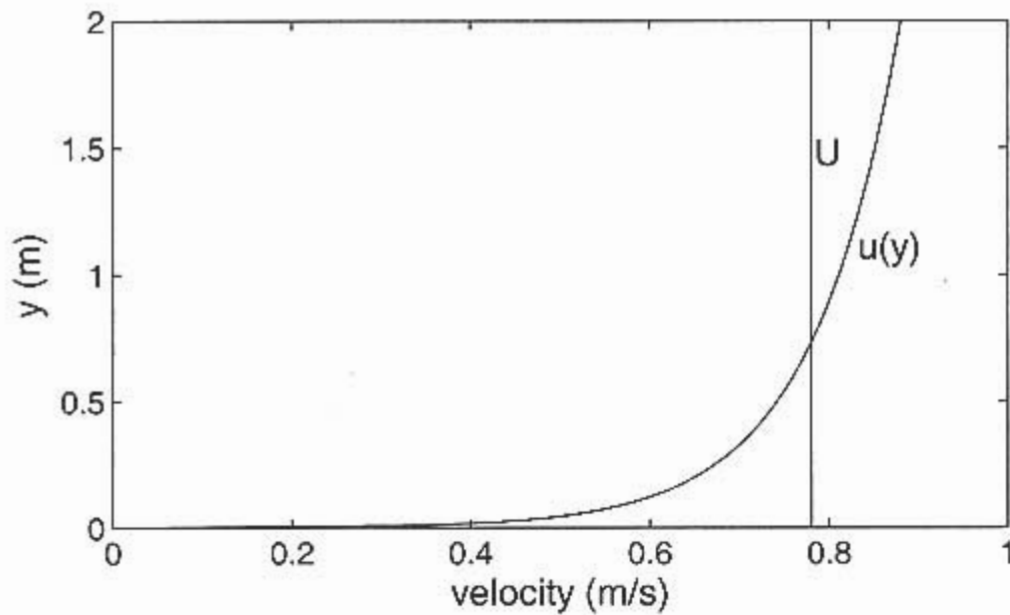
$$\frac{u_*}{\kappa} = \frac{0.04}{0.4} = 0.1 \text{ m/s}$$

$$y_0 = 3 \cdot 10^{-4} \text{ m}$$

$$u = 0.1 \ln \left( \frac{y + 3 \cdot 10^{-4}}{3 \cdot 10^{-4}} \right) \quad 0 \leq y \leq 2, \quad y \text{ in m, } u \text{ in m/s}$$

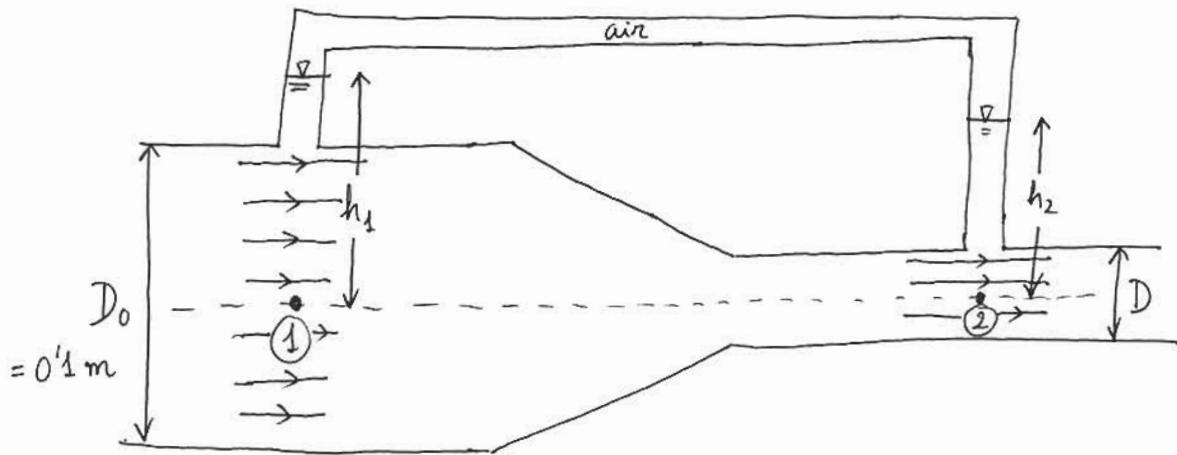
$$U = \frac{0.04}{0.4} \ln \frac{2}{e \cdot 3 \cdot 10^{-4}} = 0.780 \text{ m/s}$$

In the next page,  $u(y)$  is plotted and compared to  $U$ .



For most of the depth, the approximation  $u(y) \sim U$  is rather good. The error of this approximation at the surface is  $\frac{|U - u(2)|}{u(2)} \cdot 100\% = 11.4\%$  (relative error), and smaller than that for  $0.33 \text{ m} \leq y < 2 \text{ m}$ . However, the approximation error gets very large near the bottom, where the real velocity tends to 0 due to the no-slip boundary condition, while  $U$  remains (obviously) constant. This inaccuracy near the bottom would be critical in some cases (e.g., if we want to study erosion); otherwise,  $U$  may be taken as a first (rough) approximation of  $u(y)$ .

-PROBLEM N°5:



Continuity between (1) and (2):

$$Q = \frac{\pi D_0^2}{4} \cdot V_1 = \frac{\pi D^2}{4} \cdot V_2 \Rightarrow V_1 = \frac{D^2}{D_0^2} V_2$$

Manometer:

Note: Flow at (1) or (2) is well behaved, so pressure varies hydrostatically  $\perp$  streamlines. (See Problem No. 6)

$$\left. \begin{aligned} p_1 &= p_{\text{air}} + \rho g h_1 \\ p_2 &= p_{\text{air}} + \rho g h_2 \end{aligned} \right\} p_1 - p_2 = \rho g (h_1 - h_2)$$

Bernoulli for points (1) and (2):

$$p_1 + \frac{1}{2} \rho V_1^2 + \cancel{\rho g z_1} = p_2 + \frac{1}{2} \rho V_2^2 + \cancel{\rho g z_2} \quad (z_1 = z_2)$$

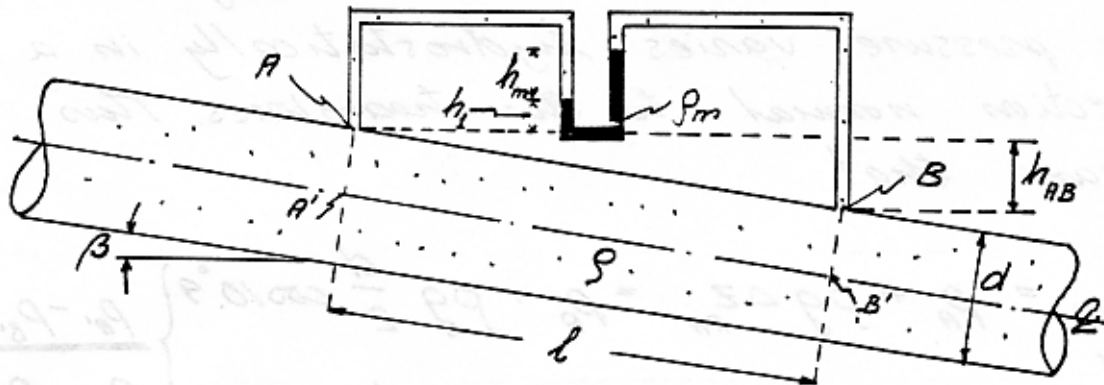
$$\frac{1}{2} \rho (V_2^2 - V_1^2) = p_1 - p_2$$

$$\frac{1}{2} \rho \left(1 - \frac{D^4}{D_0^4}\right) V_2^2 = \rho g (h_1 - h_2)$$

$$V_2 = \sqrt{\frac{2g(h_1 - h_2)}{1 - D^4/D_0^4}} = \sqrt{\frac{3.92}{1 - (D/0.1)^4}}$$

$$Q = V_2 \cdot \frac{\pi D^2}{4} = \frac{\pi}{4} \sqrt{\frac{3.92 D^4}{1 - (10D)^4}} \quad \begin{array}{l} D \text{ in } m \\ Q \text{ in } m^3/s \end{array}$$

Problem No. 6:



a) With notation indicated in figure we have starting at A:

$$p_A - h_1 \rho g - \rho_m g h_m + (h_m + h_1 + l \sin \beta) \rho g = p_B$$

$$\underline{p_A - p_B} = (\rho_m - \rho) g h_m - (l \sin \beta) \rho g =$$

$$(12.6 h_m - l \sin \beta) \rho g = (12.6 \cdot 3 - 200 \sin 10.9) \cdot 10^{-2} \cdot 10^3 \cdot 10 =$$

$$-1.9 \text{ Pa} \approx \underline{0}$$



If no flow in pipe pressure would be hydrostatic and  $p_B$  would exceed  $p_A$  by  $\rho g (l \sin \beta) = 3,780 \text{ Pa}$ , i.e., by much more than found in (a).

Thus,  $p_B$  is reduced relative to  $p_A$ , suggesting flow from  $A \rightarrow B$ .



c) Since streamlines are straight lines the pressure varies hydrostatically in a direction normal to the streamlines. This means that

$$\begin{array}{l} P_{A'} = P_A + \rho g \Delta z_{AA'} = P_A + \rho g \frac{d}{2} \cos 10.9^\circ \\ \text{and} \\ P_{B'} = P_B + \rho g \Delta z_{BB'} = P_B + \rho g \frac{d}{2} \cos 10.9^\circ \end{array} \left. \begin{array}{l} \\ \\ \end{array} \right\} \begin{array}{l} \underline{P_{A'} - P_{B'}} = \\ \underline{P_A - P_B} \end{array}$$