Problem 1 (Kang, 2001)

Let X_1 and X_2 be independent random variables denoting the two picks that are uniformly distributed over the interval $[0, a]$. Let $G(a) \equiv E[X^2] \equiv E[(\max(X_1, X_2))^2]$. Suppose $a < X_1 \leq a + \varepsilon$ and $0 \leq X_2 \leq a$. $G(a+\varepsilon)$ for this case is computed as follows:

$$G(a+\varepsilon) = E[(\max(X_1, X_2))^2] = E[X_1^2] = \int_a^{a+\varepsilon} x_1^2 \, f_{X_1}(x_1)\, dx_1 = \frac{1}{\varepsilon}\int_a^{a+\varepsilon} x_1^2\, dx_1$$
$$= \frac{1}{\varepsilon}\left[\frac{1}{3}x_1^3\right]_a^{a+\varepsilon} = a^2 + a\varepsilon + o(\varepsilon),$$

where $o(\varepsilon)$ represents higher order terms of ε satisfying $\lim_{\varepsilon \to 0} \frac{o(\varepsilon)}{\varepsilon} = 0$. Ignoring $o(\varepsilon)$, we have the following table that summarizes $G(a+\varepsilon)$'s.

Case	Probability of a case	$G(a+\varepsilon)$ given a case
$0 \leq X_1 \leq a,\ 0 \leq X_2 \leq a$	$\frac{a}{a+\varepsilon} \cdot \frac{a}{a+\varepsilon} = (\frac{a}{a+\varepsilon})^2$	$G(a)$
$a < X_1 \leq a+\varepsilon,\ 0 \leq X_2 \leq a$	$\frac{\varepsilon}{a+\varepsilon} \cdot \frac{a}{a+\varepsilon} = \frac{\varepsilon a}{(a+\varepsilon)^2}$	$a^2 + a\varepsilon$
$0 \leq X_1 \leq a,\ a < X_2 \leq a+\varepsilon$	$\frac{a}{a+\varepsilon} \cdot \frac{\varepsilon}{a+\varepsilon} = \frac{\varepsilon a}{(a+\varepsilon)^2}$	$a^2 + a\varepsilon$
$a < X_1 \leq a+\varepsilon,\ a < X_2 \leq a+\varepsilon$	$\frac{\varepsilon}{a+\varepsilon} \cdot \frac{\varepsilon}{a+\varepsilon} = (\frac{\varepsilon}{a+\varepsilon})^2$	We do not care.

Using the total expectation theorem, we obtain

$$G(a+\varepsilon) = G(a)\left(\frac{a}{a+\varepsilon}\right)^2 + (a^2 + a\varepsilon)\frac{\varepsilon a}{(a+\varepsilon)^2} + (a^2 + a\varepsilon)\frac{\varepsilon a}{(a+\varepsilon)^2} + o(\varepsilon^2)$$
$$\approx G(a)\left(\frac{a}{a+\varepsilon}\right)^2 + 2(a^2 + a\varepsilon)\frac{\varepsilon a}{(a+\varepsilon)^2}$$

From the formula of the sum of an infinite geometric series, we know

$$\frac{a}{a+\varepsilon} = \frac{1}{1+\frac{\varepsilon}{a}} = 1 - \frac{\varepsilon}{a} + \left(\frac{\varepsilon}{a}\right)^2 - \left(\frac{\varepsilon}{a}\right)^3 + \cdots$$

Ignoring higher order terms of ε, we have

$$\frac{a}{a+\varepsilon} \approx 1 - \frac{\varepsilon}{a}$$

This gives the following approximations:

$$\left(\frac{a}{a+\varepsilon}\right)^2 \approx \left(1-\frac{\varepsilon}{a}\right)^2 = 1 - \frac{2\varepsilon}{a} + \frac{\varepsilon^2}{a^2} \approx 1 - \frac{2\varepsilon}{a}$$

$$\frac{\varepsilon a}{(a+\varepsilon)^2} = \frac{\varepsilon}{a}\left(\frac{a}{a+\varepsilon}\right)^2 \approx \frac{\varepsilon}{a}\left(1-\frac{2\varepsilon}{a}\right) = \frac{\varepsilon}{a} - \frac{2\varepsilon^2}{a^2} \approx \frac{\varepsilon}{a}$$

Therefore, we can rewrite $G(a+\varepsilon)$ as

$$G(a+\varepsilon) \approx G(a)\left(1-\frac{2\varepsilon}{a}\right) + 2(a^2+a\varepsilon)\cdot\frac{\varepsilon}{a} \approx G(a)\left(1-\frac{2\varepsilon}{a}\right) + 2a\varepsilon$$

Rearranging terms, we have

$$\frac{G(a+\varepsilon)-G(a)}{\varepsilon} = -\frac{2G(a)}{a} + 2a$$

If $\varepsilon \to 0$, we have the following differential equation:

$$G'(a) = -\frac{2G(a)}{a} + 2a$$

Let $G(a) = Aa^2 + Ba + C$. Since $G(0) = 0$, we have $C = 0$. From the differential equation,

$$2Aa + B = \frac{-2Aa^2 - 2Ba}{a} + 2a = (2-2A)a - 2B$$

It gives $A = \frac{1}{2}$ and $B = 0$. Therefore

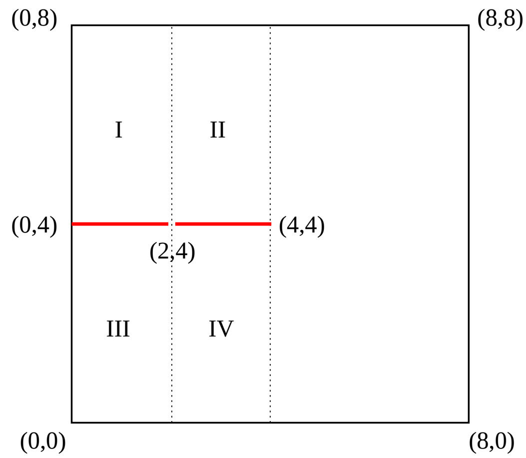
$$G(a) \equiv E[X^2] \equiv E[(\max(X_1, X_2))^2] = \frac{a^2}{2}$$

Problem 2 (Kang, 2001)

Let X_1 and X_2 denote the locations of the response vehicle and an event, respectively.

(i) The probability that the presence of the barrier increases the grid distance the vehicle must travel to the event, $P(B)$, is given by (refer to the figure below)

$$\begin{aligned} P(B) &= P(X_1 \in \mathrm{I}, X_2 \in \mathrm{III}) + P(X_1 \in \mathrm{III}, X_2 \in \mathrm{I}) + P(X_1 \in \mathrm{II}, X_2 \in \mathrm{IV}) + P(X_1 \in \mathrm{IV}, X_2 \in \mathrm{II}) \\ &= 2P(X_1 \in \mathrm{I}, X_2 \in \mathrm{III}) + 2P(X_1 \in \mathrm{II}, X_2 \in \mathrm{IV}) \\ &= 2\cdot\frac{8}{64}\cdot\frac{8}{64} + 2\cdot\frac{8}{64}\cdot\frac{8}{64} = \frac{1}{16} \end{aligned}$$



(ii) Let D be the travel distance without the barrier. We know from class

$$E[D] = E[D_x] + E[D_y] = \frac{1}{3} \times 8 + \frac{1}{3} \times 8 = \frac{16}{3}$$

Let D^e denote the extra distance the vehicle should travel due to the barrier. Let us first compute $E[D^e \mid X_1 \in \text{I}, X_2 \in \text{III}]$. There is no extra travel distance in the y axis, i.e. $E[D^e_y \mid X_1 \in \text{I}, X_2 \in \text{III}] = 0$. We also know from class that the extra travel distance in the x axis, $E[D^e_x \mid X_1 \in \text{I}, X_2 \in \text{III}]$, is $\frac{2}{3} \times 2 = \frac{4}{3}$. Hence

$$E[D^e \mid X_1 \in \text{I}, X_2 \in \text{III}] = \frac{4}{3}$$

By symmetry,

$$E[D^e \mid X_1 \in \text{III}, X_2 \in \text{I}] = \frac{4}{3}$$

Now consider $E[D^e \mid X_1 \in \text{II}, X_2 \in \text{IV}]$. As before, $E[D^e_y \mid X_1 \in \text{II}, X_2 \in \text{IV}] = 0$. To compute $E[D^e_x \mid X_1 \in \text{II}, X_2 \in \text{IV}]$, we should note that it is possible to travel through the both ends of the barrier spanning from $(2, 4)$ to $(4, 4)$. We saw in a problem set when travel is allowed through the both ends of the barrier, the extra travel distance is $\frac{1}{3}$ times the length of the barrier (refer to Problem 3.14 in the textbook). Therefore,

$$E[D^e \mid X_1 \in \text{II}, X_2 \in \text{IV}] = \frac{1}{3} \times 2 = \frac{2}{3}$$

By symmetry,

$$E[D^e \mid X_1 \in \text{IV}, X_2 \in \text{II}] = \frac{1}{3} \times 2 = \frac{2}{3}$$

$E[D^e]$ is then computed by

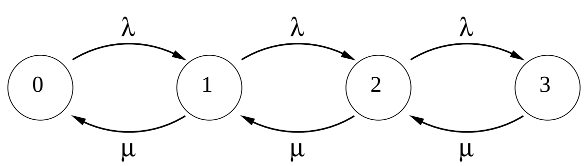
$$
\begin{aligned}
E[D^e] = {} & E[D^e \mid X_1 \in \mathrm{I}, X_2 \in \mathrm{III}]\, P(X_1 \in \mathrm{I}, X_2 \in \mathrm{III}) + \\
& E[D^e \mid X_1 \in \mathrm{III}, X_2 \in \mathrm{I}]\, P(X_1 \in \mathrm{III}, X_2 \in \mathrm{I}) + \\
& E[D^e \mid X_1 \in \mathrm{II}, X_2 \in \mathrm{IV}]\, P(X_1 \in \mathrm{II}, X_2 \in \mathrm{IV}) + \\
& E[D^e \mid X_1 \in \mathrm{IV}, X_2 \in \mathrm{II}]\, P(X_1 \in \mathrm{IV}, X_2 \in \mathrm{II}) \\
= {} & \frac{4}{3} \cdot \frac{8}{64} \cdot \frac{8}{64} + \frac{4}{3} \cdot \frac{8}{64} \cdot \frac{8}{64} + \frac{2}{3} \cdot \frac{8}{64} \cdot \frac{8}{64} + \frac{2}{3} \cdot \frac{8}{64} \cdot \frac{8}{64} = \frac{1}{16}
\end{aligned}
$$

The expected total travel distance, $E[D']$, is therefore given by

$$
E[D'] = E[D] + E[D^e] = \frac{16}{3} + \frac{1}{16} = \frac{259}{48}
$$

Problem 3 (Odoni, 2001)

(a) If the PDF of service time is negative exponential, the state transition diagram of Vincent's barbershop queueing system is given by



For the case where $\lambda = \mu$, we have the following balance equations and normalization equation:

$$
\begin{aligned}
P_0 &= P_1 \\
P_1 &= P_2 \\
P_2 &= P_3 \\
P_0 + P_1 + P_2 + P_3 &= 1
\end{aligned}
$$

Solving equations, we have $P_0 = P_1 = P_2 = P_3 = \frac{1}{4}$. The expected number of customers in the barbershop is

$$
L = 1 \times P_1 + 2 \times P_2 + 3 \times P_3 = \frac{6}{4} = 1.5
$$

(b) Suppose there are k chairs (including the barber's chair) in the barbershop, which is to be determined. The balance equations and the normalization equation in this case are given by

$$P_n = P_{n+1}, \quad \text{for } n = 0, 1, \cdots, k-1$$
$$P_0 + P_1 + \cdots + P_k = 1$$

Clearly, $P_0 = P_1 = \cdots = P_k = \frac{1}{k+1}$. To make sure that at least 92% of his prospective customers become actual customers, the probability that a new customer finds all chairs occupied, P_k, should be less than 8%.

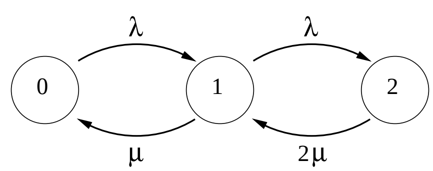
$$P_k = \frac{1}{k+1} < 0.08 \quad \Rightarrow \quad k > 11.5$$

The minimum number of chairs he will need in the shop is 12.

(c) The state transition diagram for SIRO will be the same as that for FIFO. This means that the steady-state probabilities P_n will be identical in the two cases. Then $L = \sum_{n=0}^{3} nP_n$ will the same in the two cases. Therefore, $W = \frac{L}{\lambda'} = \frac{L}{\lambda(1-P_3)}$ will be the same.

Problem 4 (Odoni, 2001)

(a) The state transition diagram of this M/M/2 queueing system is



The balance equations and the normalization equation are

$$\lambda P_0 = \mu P_1$$
$$\lambda P_1 = 2\mu P_2$$
$$P_0 + P_1 + P_2 = 1$$

$P_1 = \frac{\lambda}{\mu} P_0 = \rho P_0$. $P_2 = \frac{\lambda}{2\mu} P_1 = \frac{1}{2}\rho P_1 = \frac{1}{2}\rho^2 P_0$. Using the normalization equation,

$$P_0 + \rho P_0 + \frac{1}{2}\rho^2 P_0 = P_0\left(1 + \rho + \frac{1}{2}\rho^2\right) = 1 \quad \Rightarrow \quad P_0 = \frac{1}{1 + \rho + \frac{1}{2}\rho^2}$$

The expected number of men who are busy serving a customer at any given time is given by

$$1 \times P_1 + 2 \times P_2 = \frac{\rho}{1 + \rho + \frac{1}{2}\rho^2} + \frac{\rho^2}{1 + \rho + \frac{1}{2}\rho^2} = \frac{\rho + \rho^2}{1 + \rho + \frac{1}{2}\rho^2}$$

(b) Using the data collected, we have the following equation:

$$\frac{\rho+\rho^2}{1+\rho+\frac{1}{2}\rho^2} = \frac{8,000}{10,000} = 0.8 \quad \Rightarrow \quad 0.8+0.8\rho+0.4\rho^2 = \rho+\rho^2$$

$$\Rightarrow \quad 0.6\rho^2+0.2\rho-0.8=0$$

$$\Rightarrow \quad \rho^2+\frac{1}{3}\rho-\frac{4}{3}=0$$

It gives $\rho = 1$ (the other root, $-\frac{4}{3}$, is meaningless). Note that the actual arrival rate of customers is $\lambda' = \lambda(1-P_2)$. Since 40,000 customers received service during 10,000 hours,

$$\lambda(1-P_2) = \frac{40,000}{10,000} = 4$$

Since $\rho = 1$, we have $P_2 = \frac{\frac{1}{2}\rho^2}{1+\rho+\frac{1}{2}\rho^2} = \frac{1}{5}$. Therefore

$$\lambda = \frac{4}{(1-P_2)} = \frac{4}{4/5} = 5$$

The number of customers lost during these 10,000 hours is

$$\lambda P_2 \times 10,000 = 5 \times \frac{1}{5} \times 10,000 = 10,000$$