The key to this problem is to understand random incidence. The best way to approach the problem is to view the interval h in which Mendel arrives as two pieces. These are:

1. The length of time(t) from the last bus till Mendel arrives
2. The length of time(u) from when Mendel arrives to the next bus.

Thus $E[u]+E[t]=E[h]$

Clearly the sum of these two expected values is the expected value of time between busses for a randomly chosen interval. Let's answer the second one first. Since arrivals are Poisson and the Poisson process is memoryless the time from when Mendel arrives to the next bus is clearly a Poisson random variable with parameter λ.

Thus $E[u]=\frac{1}{\lambda}$.

Now in order to evaluate quantity one, we have to realize that a Poisson process is reversible in time. Thus we can view the variable u alternatively as the time between when Mendel arrives till the previous bus arrived. Hence we can look forward and backwards in time. Thus looking at t this way it is clear that u is also a Poisson variable with parameter λ.

Thus $E[t]=\frac{1}{\lambda}$.

Combining these two we can find the expected value of h.

$$E[h]=E[t]+E[u]=\frac{1}{\lambda}+\frac{1}{\lambda}=\frac{2}{\lambda}$$

Alternately, we can use our formula for the mean time between Mendel's arrival and the end of his interval, based on the mean and variance of the distribution of intervals. Then, knowing that on average he arrives in the middle of the interval, we double the calculated result to get the mean total interval. Once again, the answer is $2/\lambda$.

2.

We let:

$G(a)=E[xy]$

Using Crofton's method we calculate:

$G(a+\varepsilon)$ using cases:

Case	P(Case)	E[xy|Case]
x,y<=a	(a/(a+e))^2	G(a)
x>a,y<=a	(e/(a+e))*(a/(a+e))	(a/2)*(a+(e/2))
x<=a,y>a	(e/(a+e))*(a/(a+e))	(a/2)*(a+(e/2))
x,y>a	(e/(a+e))^2	We Do Not Care

A key fact we used when calculating E[xy|Case 2] and E[xy|Case 3] is that since x and y are independent we know E[xy]=E[x]*E[y].

Thus:

$$G(a+e) = (\frac{a}{a+e})^2 G(a) + 2(\frac{e}{a+e})(\frac{a}{a+e})(\frac{a}{2})(a+\frac{e}{2}) + o(e^2)$$

Since we know:

$$\frac{a}{a+e} \approx 1 - \frac{e}{a}$$

$$\frac{e}{a+e} \approx \frac{e}{a}$$

$$(1-\frac{e}{a})^2 \approx 1 - \frac{2e}{a}$$

Using these approximations we get:

$$G(a+e) \approx (1-\frac{e}{a})^2 G(a) + 2(\frac{e}{a})(a+\frac{e}{2})(1-\frac{e}{a})(\frac{a^2}{2}+\frac{ae}{4}) + o(e^2)$$

Simplifying we get:

$$G(a+e) \approx G(a) - \frac{2e}{a}G(a) + ea + o(e^2)$$
$$\frac{G(a+e)-G(a)}{e} = -\frac{2}{a}G(a) + a$$
$$G'(a) = -\frac{2}{a}G(a) + a$$

We now solve this differential equation using the guess $G(a) = ka^2$

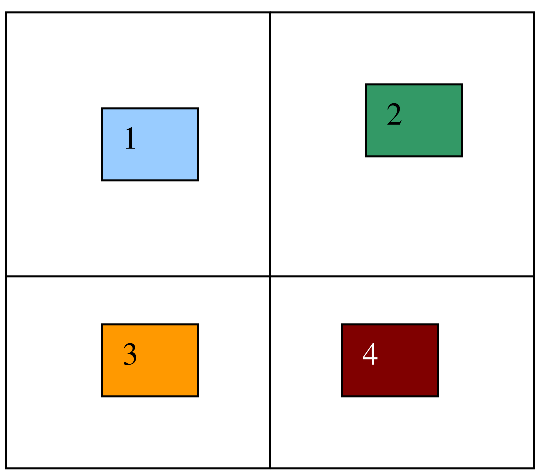
Plugging this into the above we get:

$$2ka = \frac{-2ka^2}{a} + a$$
$$2ka = -2ka + a$$
$$4ka = a$$
$$k = \frac{1}{4}$$
$$G(a) = \frac{a^2}{4}$$

Of course, we didn't really need Crofton's method in this case. Each of the quantities has mean a/2 and they are independent, so the mean of their product is $a^2/4$. But we wanted you to find the answer using Crofton's method, as almost everyone did so successfully.

3.

(i)



The barrier clearly divides the city into 4 quadrants. The probability an point occurs in a quadrant is proportional to the area. Thus

$P(x = Quadrent1) = \frac{3}{10}$

$P(x = Quadrent2) = \frac{3}{10}$

$P(x = Quadrent3) = \frac{2}{10}$

$P(x = Quadrent4) = \frac{2}{10}$

Thus the probability that the barrier will affect the travel distance between two points(x,y) is the same as the probability the two points are in different quadrants that are not diagonally opposite each other. Thus we calculate this as follows noting that $P(x = Q1, y = Q2) = P(x = Q2, y = Q1)$ which simplifies the calculation

$$P(BarrierIncresesDistance) = 2P(x = Q1, y = Q2) + 2P(x = Q1, y = Q4) + 2P(x = Q2, y = Q3) + 2P(x = Q3, y = Q4)$$

Where by independence we know: $P(x = Q1, y = Q2) = P(x = Q1)P(y = Q2)$

Thus:

$$P(BarrierIncresesDistance) = 2 * \frac{3}{10} * \frac{3}{10} + 2 * \frac{3}{10} * \frac{2}{10} + 2 * \frac{3}{10} * \frac{2}{10} + 2 * \frac{2}{10} * \frac{2}{10} = \frac{1}{2}$$

(ii)

It is clear from part (i) that only one of the barriers can increase the travel distance between two points. Thus clearly the longest part of a barrier is the place where travel distances can be increased the most. This occurs at the upper part of the barrier that runs parallel to the y axis. From class we know the maximum the travel distance can be increased is twice the length of this barrier(i.e. one point located in 1 along the top of the city, one point in 2 along the top of the city). Thus the maximum increase in distance the barrier can cause is 2*3/5=6/5.

(iii)

We first note from class the expected x distance between two random points chosen is 1/3, similarly the expected y distance between two random points chosen is 1/3. Thus the expected travel distance without the barrier is 2/3. The expected travel distance given the barrier is the sum of two quantities:

$$E[d \mid Barrier] = E[d] + E[ExtraDistance \mid Barrier]$$

$$E[d] = \frac{2}{3}$$

Using cases we can calculate the expected extra distance caused by the barrier:

Case (a in Qi, b in Qj)	P(Case)	E[Extra Distance|Case]
(1,2)	0.09	0.40000
(2,1)	0.09	0.40000
(1,4)	0.06	0.33333
(4,1)	0.06	0.33333
(2,3)	0.06	0.33333
(3,2)	0.06	0.33333
(3,4)	0.04	0.26667
(4,3)	0.04	0.26667

Where we note:

- The expected extra distance for the first two cases occurs due to the barrier parallel, to the y axis. We expect the points y coordinate to be halfway between 2/5 and 1 thus being at y=7/10. Thus the extra travel distance is 2*(7/10 – 4/10)-(1/3)*(3/5) = 2/5.

- The expected extra distance for cases three and four occurs due to the barrier parallel, to the x axis. We expect the points x coordinate to be halfway between 0 and 1/2 thus being at x=1/4. Thus the extra travel distance is 2*(2/4 – 1/4)-(1/3)*(1/2) = (1/3)

- Cases five and six are the same as three and four except the x coordinate of each point is uniform on [1/2,1] as opposed to [0,1/2]. Thus the expected extra travel distance is again 1/3.

- The expected extra distance for the cases seven and eight occurs due to the barrier parallel, to the y axis. We expect the points y coordinate to be halfway between 0 and 2/5 thus being at y=1/5. Thus the extra travel distance is 2*(2/5 – 1/5)-(1/3)*(2/5) = 4/15.

Thus we get:

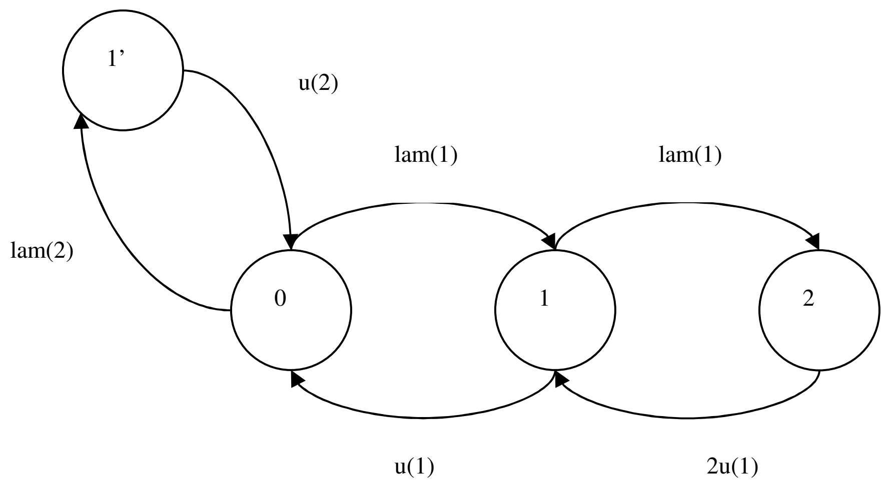
$$E[d \mid Barrier] = 0.66666 + 2(0.09 * 0.4) + 4(0.06 * 0.3333) + 2(0.04 * .2667) = 0.84$$

4.

a. This is a finite-state system which excess demands lost, so it reaches steady state no matter what the values of $\lambda_1, \mu_1, \lambda_2$ and μ_2 are.

b. Define state 1' as the state in which both servers are occupied by a Type 2 customer.

Then:



Therefore:

$$P_{1'}\mu_2 = P_0\lambda_2 \Rightarrow P_{1'} = P_0(\frac{\lambda_2}{\mu_2})$$

$$P_0\lambda_1 = P_1\mu_1 \Rightarrow P_1 = P_0(\frac{\lambda_1}{\mu_1})$$

$$P_2 * 2\lambda_1 = P_1\lambda_1 \Rightarrow P_2 = P_1(\frac{\lambda_1}{2\mu_1}) = P_0 * \frac{1}{2}(\frac{\lambda_1}{\mu_1})^2$$

and

$$P_0 = \frac{1}{1 + \frac{\lambda_1}{\mu_1} + \frac{\lambda_2}{\mu_2} + \frac{1}{2}(\frac{\lambda_1}{\mu_1})^2}$$

c.

Fraction of Type 1 lost $= P_{1'} + P_2 = 1 - P_0 - P_1$

Fraction of Type 2 lost $= 1 - P_0$

d.

$L = 1 * P_1 + 2 * P_2$