Lecture Notes on Fluid Dynamics
(1.63J/2.21J)
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[Ref] Lamb : *Hydrodynamics*

2.6 Oseen's improvement for slow flow past a cylinder

2.6.1 Oseen's criticism of Stokes' approximation

Is Stokes approximtion always good for small Reynolds number?

The exact Navier Stokes equations read

$$\nabla \cdot \vec{q} = 0 \tag{2.6.1}$$

$$\vec{q} \cdot \nabla \vec{q} = -\nabla p + \nu \nabla^2 \vec{q}. \tag{2.6.2}$$

Let us estimate the error of Stokes approximation where the velocity field is given by

$$q_y = W \cos\theta \left(1 + \frac{a^3}{2r^3} - \frac{3a}{2r}\right), \quad q_z = -W \sin\theta \left(1 - \frac{a^3}{4r^3} - \frac{3a}{4r}\right). \tag{2.6.3}$$

In the far field $r/a \gg 1$, the viscous stress is dominated by the last term

$$\nabla^2 \vec{q} = O\left(\frac{a^3}{r^3}\right).$$

The inertia term is dominated by

$$W \frac{\partial \vec{q}}{\partial z} \sim O\left(\frac{a^2}{r^2}\right).$$

Hence the error is their ratio

$$\frac{W \frac{\partial \vec{q}}{\partial z}}{\nu \nabla^2 \vec{q}} = O\left(\frac{r}{a}\right),$$

which becomes unbounded for $r/a \gg 1$. Thus inertia cannot be ignored in the far field.

The difficulty is severe for the two-dimensional flow past a cylinder. By taking the curl, Stokes equation gives

$$\nabla^2 \vec{\zeta} = 0.$$

Since the body is a source of vorticity, $\vec{\zeta}$ would become unbounded logarithmically for large r/a. This is certainly unphysical and is known as Stokes' paradox.

In view of the increasing importance of inertia in the far field, Oseen suggests that the linear approximation of the inertia term, which is of dominant importance in the far field, and not in the near field, be added. Let $\vec{q} = W\vec{k} + \vec{q'}$ the Oseen equations are:

$$\nabla \cdot \vec{q'} = 0 \tag{2.6.4}$$

$$W\frac{\partial \vec{q'}}{\partial z} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \vec{q'}. \tag{2.6.5}$$

The added linear inertia term is of dominant importance in the far field and not in the near field.

We shall demonstrate the Oseen approximation for the two-dimensional case of a circular cylinder.

2.6.2 Oseen's theory for a circular cylinder

Because of (2.6.4), the pressure is still harmonic

$$\nabla^2 p = 0. \tag{2.6.6}$$

Now the velocity can be expressed as the sum of a potential part $\nabla\phi$ and a solenoidal part $\vec{q''}$, where

$$\vec{q'} = \nabla\phi + \vec{q''} \tag{2.6.7}$$

It follows that

$$\nabla^2 \phi = 0 \tag{2.6.8}$$

and vorticity is only associated with $\vec{q''}$. Substituting Eq. () into Eq. () and setting

$$p = p_\infty = -\rho W \frac{\partial \phi}{\partial z} \tag{2.6.9}$$

we get

$$W\frac{\partial \vec{q''}}{\partial z} = \nu \nabla^2 \vec{q''}, \tag{2.6.10}$$

which implies

$$W\frac{\partial \vec{\zeta}}{\partial z} = \nu \nabla^2 \vec{\zeta}, \tag{2.6.11}$$

where $\vec{\zeta} = \nabla \times \vec{q''}$. To solve for $\vec{q''}$ let us introduce a vector potential $\vec{k}\sigma$

$$\vec{\zeta} = \nabla \times (\vec{k}\sigma). \tag{2.6.12}$$

Now

$$\nabla \times \vec{\zeta} = \nabla \times (\nabla \times \vec{k}\sigma) = \nabla(\nabla \cdot (\vec{k}\sigma)) - \vec{k}\nabla^2\sigma = \nabla \frac{\partial \sigma}{\partial z} - \vec{k}\nabla^2\sigma.$$

On the other hand

$$\nabla \times \vec{\zeta} = \nabla \times (\nabla \times \vec{q''}) = \nabla(\nabla \cdot \vec{q''}) - \nabla^2 \vec{q''} = -\nabla^2 \vec{q''} = -\frac{W}{\nu}\frac{\partial \vec{q''}}{\partial z}.$$

after using (2.6.10). It follows by equating the two preceding equations,

$$-\frac{\overline{W}}{\nu}\frac{\partial \vec{q''}}{\partial z} = \nabla \frac{\partial \sigma}{\partial z} - \vec{k}\nabla^2 \sigma. \tag{2.6.13}$$

Now (2.6.12) does not define σ uniquely. Let us impose a condition

$$W\frac{\partial \sigma}{\partial z} = \nu \nabla^2 \sigma. \tag{2.6.14}$$

Then we can integrate to get

$$-\frac{W}{\nu}\vec{q''} = \nabla\sigma - \frac{W}{\nu}\vec{k}\,\sigma, \tag{2.6.15}$$

hence,

$$\vec{q''} = \vec{k}\,\sigma - \frac{\nu}{W}\nabla\sigma. \tag{2.6.16}$$

In polar coordinates the velocity components are

$$q''_r = \sigma \cos\theta - \frac{\nu}{W}\frac{\partial \sigma}{\partial r} \tag{2.6.17}$$

$$q''_\theta = -\sigma \sin\theta - \frac{\nu}{W}\frac{1}{r}\frac{\partial \sigma}{\partial \theta}. \tag{2.6.18}$$

The total disturbance is

$$q'_r = \frac{\partial \phi}{\partial r} + \sigma\cos\theta - \frac{\nu}{W}\frac{\partial \sigma}{\partial r} \tag{2.6.19}$$

$$q'_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} - \sigma\sin\theta - \frac{\nu}{W}\frac{1}{r}\frac{\partial \sigma}{\partial \theta}. \tag{2.6.20}$$

It is convenient to rearrange (2.6.14) by introducing

$$\sigma(y,z) = e^{Wz/2\nu}\bar{\sigma}(y,z) \tag{2.6.21}$$

so that

$$\nabla^2\bar{\sigma} - \left(\frac{W}{2\nu}\right)^2 \bar{\sigma} = 0, \tag{2.6.22}$$

We now introduce dimensionless variables

$$\phi = Wa\,\Phi, \quad \sigma = W\Sigma, \quad \bar{\sigma} = W\bar{\Sigma}, \quad \vec{q'} = W\vec{Q}, \quad r = aR. \tag{2.6.23}$$

Then the governing equatios are

$$\nabla^2\Phi = 0 \tag{2.6.24}$$

and

$$(\nabla^2 - \epsilon^2)\Phi = 0 \qquad \epsilon = \frac{Wa}{2\nu} \tag{2.6.25}$$

The velocity components are

$$\begin{aligned} Q_r &= \frac{\partial \Phi}{\partial R} + \sigma \cos\theta - \frac{1}{2\epsilon}\frac{\partial \Phi}{\partial R} \\ Q_\theta &= \frac{1}{R}\frac{\partial \Phi}{\partial R} - \phi \sin\theta - \frac{1}{2\epsilon}\frac{1}{R}\frac{\partial \Phi}{\partial \theta}. \end{aligned} \tag{2.6.26}$$

The boundary conditions are

$$Q_R + \cos\theta = 0 \quad R = 1 \tag{2.6.27}$$

$$Q_\theta - \sin\theta = 0 \quad R = 1. \tag{2.6.28}$$

The general solution to (2.6.24) and (2.6.25) is

$$\Phi = A_0 \ln R + A_1 \frac{\cos\theta}{R} \tag{2.6.29}$$

and

$$\sigma = C_0 K_0(\epsilon R) \tag{2.6.30}$$

We shall determine the coefficients $A_0\, A_1$ and C_0 approximately for small $\epsilon = Re/2$.

On the cylinder $R = 1$, we use the approximation for $\epsilon R \ll 1$

$$\begin{aligned} K_0(\epsilon R) &\cong -\left(\sigma + \frac{1}{2}\ln\frac{\epsilon R}{2}\right) I_0(\epsilon R) + \frac{\epsilon^2 R^2}{2} + \cdots \\ &\cong -\left(\sigma + \frac{1}{2}\ln\frac{\epsilon R}{2}\right) + ... \end{aligned} \tag{2.6.31}$$

$$\begin{aligned} \sigma &\cong C_0\, e^{\epsilon R\cos\theta} K_0(\epsilon R) \\ &\cong -C_0(1 + \epsilon R\cos\theta + \cdots)(\sigma + \frac{1}{2}\ln\frac{\epsilon R}{2} + \cdots) \end{aligned} \tag{2.6.32}$$

Details of calculation for small ϵR are given for convenience

$$\frac{\partial \Phi}{\partial R} = \frac{A_0}{R} - \frac{A_1}{R^2}\cos\theta$$

$$\frac{1}{R}\frac{\partial \Phi}{\partial \theta} = -\frac{A_1}{R}\sin\theta\,|_1$$

$$\sigma\cos\theta\,|_1 \cong -C_0(1 + \epsilon R\cos\theta + \cdots)\left(\sigma + \ln\frac{\epsilon R}{2}\right)\cos\theta$$

$$-\sigma\sin\theta\,|_1 \cong +C_0(1 + \epsilon R\cos\theta + \cdots)\left(\sigma + \ln\frac{\epsilon R}{2}\right)\sin\theta$$

$$
\begin{aligned}
-\frac{1}{2\epsilon}\frac{\partial\sigma}{\partial R} &\cong +\frac{1}{2\epsilon}\left[C_0\,\epsilon\,\cos\,\theta(1+\epsilon R\,\cos\,\theta)\left(\sigma+\ln\,\frac{\epsilon R}{2}\right)+\cdots\right] \\
&\quad +\frac{1}{2\epsilon}\,C_o(1+\epsilon R\,\cos\,\theta+\cdots)\left(-\frac{1}{R}\right)+\cdots \\
-\frac{1}{2\epsilon}\,\frac{1}{R}\,\frac{\partial\sigma}{\partial\theta} &\cong +\frac{1}{2\epsilon}\,\frac{1}{R}\left[C_0(-\epsilon R\,\sin\,\theta)(1+\epsilon\,\cos\,\theta)\left(\sigma+\ln\,\frac{\epsilon}{2}\right)\right]
\end{aligned}
$$

On the cyolinder $R=1$ condition on the radial velocity requires that

$$
\begin{aligned}
Q_R = -\cos\,\theta = A_0 - A_1\,\cos\,\theta - C_0(1+\epsilon R\,\cos\,\theta)\left(\sigma+\ln\,\frac{\epsilon R}{2}\right)\,\cos\,\theta \\
-\frac{1}{2\epsilon}\left[-C_0\,\epsilon\,\cos\,\theta(1+\epsilon\,\cos\,\theta)\left(\sigma+\ln\,\frac{\epsilon}{2}\right)+\cdots\right] \\
-\frac{1}{2\epsilon}\,C_0(1+\epsilon R\,\cos\,\theta)\left(-\frac{1}{R}\right)+\cdots
\end{aligned}
$$

Neglecting $O(\epsilon\,\ln\,\epsilon)$ and equating coefficients of constant terms an $\cos\theta$ separately,

$$A_0+\frac{C_0}{2\epsilon}=0 \tag{2.6.33}$$

$$-A_1-\left(1-\frac{1}{2}\right)C_0\left(\sigma+\ln\,\frac{\epsilon}{2}\right)+\frac{C_0}{2}=-1 \tag{2.6.34}$$

From the condition $Q_\theta=\sin\,\theta$, we get

$$+A_1+1-\frac{C_0}{2}\left(\sigma+\ln\,\frac{\epsilon}{2}\right)=0 \tag{2.6.35}$$

The final results are :

$$C_0=\frac{-2}{\frac{1}{2}-\left(\sigma+\ln\,\frac{\epsilon}{2}\right)}. \tag{2.6.36}$$

$$A_0=+\frac{1}{\epsilon}\,\frac{1}{\frac{1}{2}-\left(\sigma+\ln\,\frac{\epsilon}{2}\right)} \tag{2.6.37}$$

and

$$A_1=\frac{\frac{1}{2}}{\frac{1}{2}-\left(\sigma+\ln\,\frac{\epsilon}{2}\right)}=\frac{1}{4}C_0 \tag{2.6.38}$$

As a physical deduction, we leave it as an exercise to show that

$$\text{drag force/length }=\frac{4\pi\mu W}{\frac{1}{2}-\sigma-\ln\,\frac{\epsilon}{2}} \tag{2.6.39}$$

so that the drag coefficient is

$$C_D=\frac{D}{\left(\frac{1}{2}\rho W^2\right)2a}=\frac{2\pi}{\epsilon}\,\frac{1}{\frac{1}{2}-\sigma-\ln\,\frac{\epsilon}{2}} \tag{2.6.40}$$

In the far wake $\epsilon R \gg 1$

$$K_0(\epsilon R) \cong \sqrt{\frac{\pi}{2\epsilon R}}\, e^{-\epsilon R} \left(1 - \frac{1}{8\epsilon R} + \cdots\right) \tag{2.6.41}$$

the rotational part is

$$C_0\, e^{\epsilon R \cos\theta} \sqrt{\frac{\pi}{2\epsilon R}}\, e^{-\epsilon R} = C_0 \sqrt{\frac{\pi}{2\epsilon R}}\, e^{-\epsilon R(1-\cos\theta)} \tag{2.6.42}$$

There is significant contribution only in the region $\theta \ll 1$. Since $1 - \cos\theta \cong \frac{\theta^2}{2}$,

$$\begin{aligned} Q_R &= C_0 \sqrt{\frac{\pi}{2\epsilon R}}\, e^{-\epsilon R(1-\cos\theta)} + \frac{Q_0}{R} \\ &\cong C_0 \sqrt{\frac{\pi}{2\epsilon X}}\, e^{-\epsilon R\theta^2/2} + \frac{A_0}{R} = C_0 \sqrt{\frac{\pi}{2\epsilon X}}\, e^{-\epsilon Y^2/2X} + \frac{A_0}{R}. \end{aligned} \tag{2.6.43}$$

the wake is therefore parabolic in shape. In the far field, the potential part s emits fluid isotropically as a source at the discharge rate $(2\pi A_0)$. On the other hand the rotational wake is a fluid sink with the influx rate

$$2C_0 \int_0^\infty \sqrt{\frac{\pi}{2\epsilon R}}\, e^{-\epsilon R\theta^2/2}\, R\, d\theta = \frac{2C_0\sqrt{\pi}}{\epsilon} \int_0^\infty e^{-\eta^2}\, d\eta = \frac{\pi C_0}{\epsilon} = -2\pi A_0 \tag{2.6.44}$$

in view of (2.6.33). Mass is conserved.