12.005 Lecture Notes 7

Newton's second law

For a point mass

$$\underset{\sim}{F} = m\underset{\sim}{a}$$

We can obtain $\underset{\sim}{F}$ from a free-body diagram – e.g., pendulum.

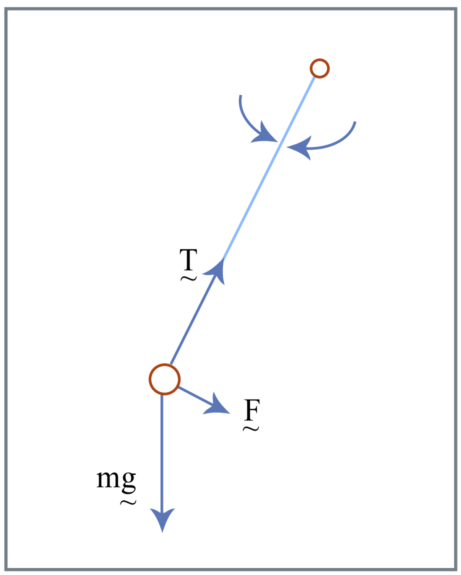

Figure 7.1

Figure by MIT OCW.

Both $\underset{\sim}{F}$ and $\underset{\sim}{a}$ are vectors.

$$F_i = ma_i \qquad i = 1, 2, 3$$

For a continuum, we can also construct a free-body diagram. It's easiest to do this component-by- component.

Consider the following figure.

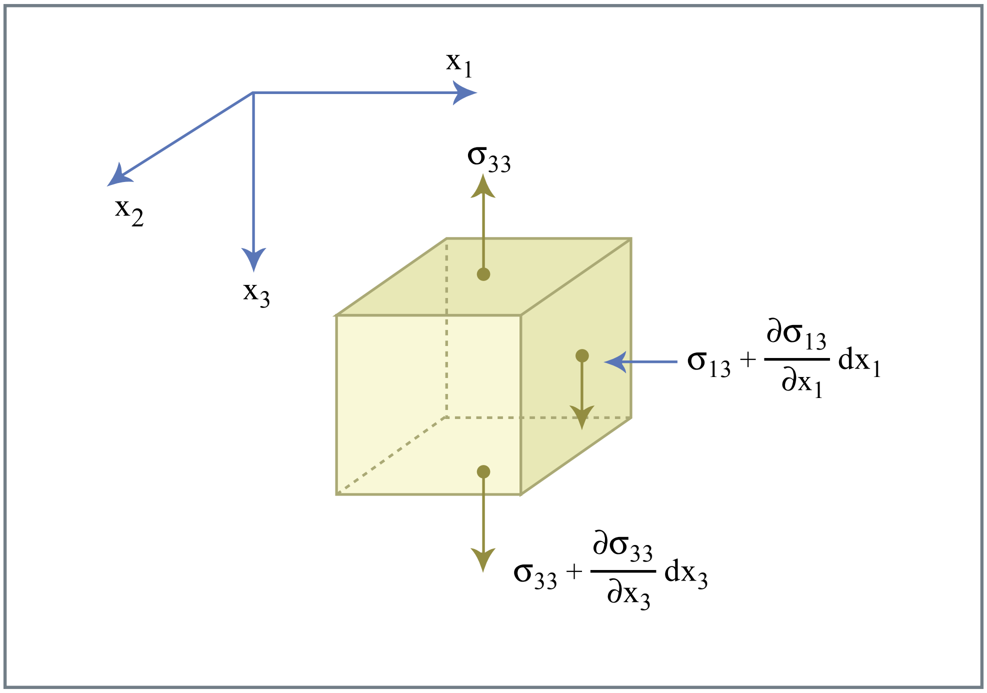

Figure 7.2

Figure by MIT OCW.

Consider forces on faces.

face	Traction (T_3)	area
left	$-\sigma_{13}$	$dx_2 dx_3$
right	$\sigma_{13} + \frac{\partial \sigma_{13}}{\partial x_1} dx_1$	$dx_2 dx_3$
back	$-\sigma_{23}$	$dx_1 dx_3$
front	$\sigma_{23} + \frac{\partial \sigma_{23}}{\partial x_2} dx_2$	$dx_1 dx_3$
top	$-\sigma_{33}$	$dx_1 dx_2$
bottom	$\sigma_{33} + \frac{\partial \sigma_{33}}{\partial x_3} dx_3$	$dx_1 dx_2$

Force (F_3) in x_1 direction: $\frac{\partial \sigma_{13}}{\partial x_1} dx_1 dx_2 dx_3$

Force (F_3) in x_2 direction: $\frac{\partial \sigma_{23}}{\partial x_2} dx_1 dx_2 dx_3$

Force (F_3) in x_3 direction: $\frac{\partial \sigma_{33}}{\partial x_3} dx_1 dx_2 dx_3$

Body force: $\rho b_3 dx_1 dx_2 dx_3$

Combining

$$\frac{\partial \sigma_{13}}{\partial x_1} dx_1 dx_2 dx_3 + \frac{\partial \sigma_{23}}{\partial x_2} dx_1 dx_2 dx_3 + \frac{\partial \sigma_{33}}{\partial x_3} dx_1 dx_2 dx_3 + \rho b_3 dx_1 dx_2 dx_3 = \rho a_3 dx_1 dx_2 dx_3$$

Dividing through by $\delta V = dx_1 dx_2 dx_3$

$$\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + \rho b_3 = \rho a_3$$

Similar analysis gives

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + \rho b_1 = \rho a_1$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + \rho b_2 = \rho a_2$$

or

$$\sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i = \rho a_i \quad i = 1, 2, 3$$

or

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i = \rho a_i \quad i = 1, 2, 3 \quad \text{(Einstein summation)}$$

or

$$\underset{\sim}{\nabla} \cdot \underset{\approx}{\sigma} + \rho \underset{\sim}{b} = \rho \underset{\sim}{a}$$

This three equations are known as the equilibrium equations, if $a_i = 0$.

To avoid accelerations, the stress tensor must satisfy equilibrium.

Aside – By the continuum mechanics definition,

positive $\sigma_{11} \Rightarrow$ extension

negative $\sigma_{11} \Rightarrow$ compression

Geologist often use the opposite convention – beware!