Course 12.520, Geodynamics
Prof. Brad Hager
Lecture 26: Corner Flow

Why aren't subducted slabs curled over or vertical?

What is the flow pattern beneath oceanic ridges?

Does constant thickness basaltic crust make sense?

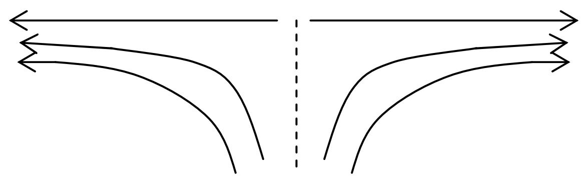


Figure 26.1

<u>Corner flow models:</u>

Batchelor – Introduction to fluid mechanics
McKenzie – GJRAS <u>18</u>, 1-32, 1969
Stevenson & Turner, Nature, <u>270</u>, 334-336, 1977
Tovish et al, JGR <u>83</u>, 5892, 1978
McAdoo, JGR, <u>87</u>, 8684, 1982

Slab model:

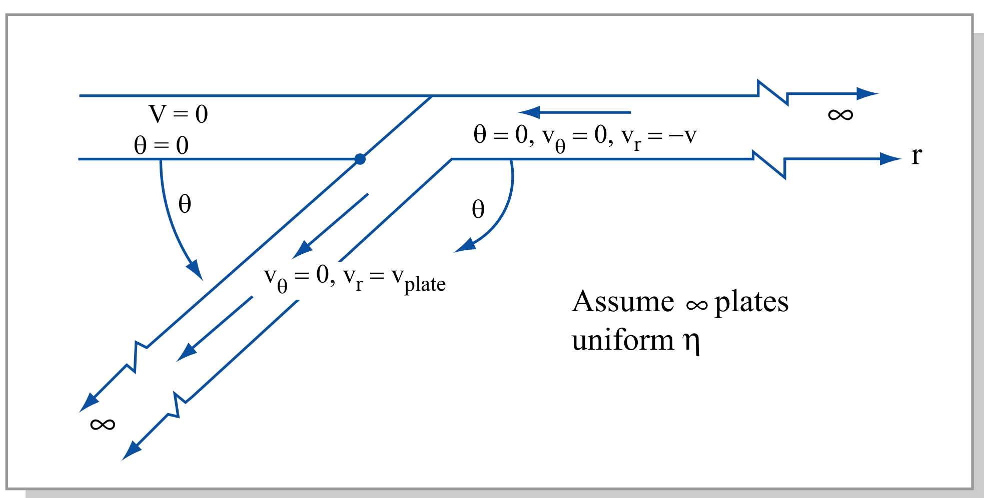


Figure by MIT OCW.

Stokes equation

$$0 = \eta \nabla^2 \underset{\sim}{v} + \rho \nabla U - \nabla p$$

$$0 = \nabla \cdot v$$

Use vorticity

$$\omega = \nabla \times \underset{\sim}{v}$$

Take curl of Stokes equation (heading toward 4th order equation)

$$0 = \eta \nabla \times \nabla^2 v + \nabla \times \left(\rho \nabla U - \nabla p \right)$$

Recall $\nabla \times \left(\nabla \varphi \right) = 0,\ \nabla \cdot \left(\nabla \times \underset{\sim}{A} \right) = 0$

Assume ρ constant

Recall $\nabla^2 \underset{\sim}{v} = \nabla \left(\nabla \cdot \underset{\sim}{v} \right) - \nabla \times \nabla \times v$

$$
\begin{aligned}
0 &= -\eta\left(\nabla\times\nabla\times\nabla\times v\right)\\
&= -\eta\left(\nabla\times\nabla\times\underset{\sim}{\omega}\right)\\
&= \eta\nabla^2\underset{\sim}{\omega}
\end{aligned}
$$

Next – define stream function $\underset{\sim}{\psi}$ by $\underset{\sim}{v} = \nabla\times\underset{\sim}{\psi}$

Automatically satisfies $\nabla\cdot\underset{\sim}{v} = 0$

Then $\underset{\sim}{\omega} = \nabla\times\nabla\times\psi = -\nabla^2\underset{\sim}{\psi}$

For our purposes, consider 2-D flow only

$$\underset{\sim}{\psi} = \left(0, 0, \psi\right)$$

r, θ, z

Must solve scalar equation

$$\nabla^2\left(\nabla^2\psi\right) = 0$$

$\nabla^4\psi = 0 \quad \leftarrow$ Biharmonic equation [Recall Airy stress function]

To help in guessing solution, recall

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}$$

$$v_\theta = -\frac{\partial\psi}{\partial r}$$

$$\tau_{rr} = -p + 2\eta\frac{\partial v_r}{\partial r}$$

$$\tau_{\theta\theta} = -p + 2\eta\left(\frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_r}{r}\right)$$

$$\tau_{r\theta} = \eta\left(\frac{1}{r}\frac{\partial v_r}{\partial\theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right)$$

$$\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial z^2}$$

Let's try solutions

$$\psi = R\left(r\right)\Theta\left(\theta\right)$$

We will want solutions with v_r independent of r

Try $R\left(r\right) = r$

$\psi = r\Theta$

Plugging in $\Rightarrow \dfrac{d^4\Theta}{d\theta^4} + 2\dfrac{d^2\Theta}{d\theta^2} + \Theta = 0,$

or $\Theta'''' + 2\Theta'' + \Theta = 0$

Solution:

$$\boxed{\Theta = A\sin\theta + B\cos\theta + C\theta\sin\theta + D\theta\cos\theta}$$

Now – match boundary conditions

$$v_r = \Theta' = A\cos\theta - B\sin\theta + C\left(\sin\theta + \theta\cos\theta\right) + D\left(\cos\theta - \theta\sin\theta\right)$$

$$v_\theta = -\Theta$$

Slab problem –

Break into 2 regions

Back-arc region		Fore-arc region	
$\theta = 0$	$v_\theta = 0 = B$ $v_r = 0 = A + D$	$\theta = 0$	$v_\theta = 0$ $v_r = -v$
$\theta = \theta_B$	$v_\theta = 0 = A\sin\theta_B + C\theta_B\sin\theta_B + D\theta_B\cos\theta_B$ $v_r = v$ $= A\cos\theta_B + C\left(\sin\theta_B + \theta_B\cos\theta_B\right) + D\left(\cos\theta_B - \theta_B\sin\theta_B\right)$	$\theta = \theta_F$	$v_\theta = 0$ $v_r = v$

Need to solve 3 equations, 3 unknowns

Solutions

Back-arc

$$\psi = \frac{rv\left[\left(\theta_b - \theta\right)\sin\theta_b\sin\theta - \theta_b\theta\sin\left(\theta_b - \theta\right)\right]}{\theta_b^{\,2} - \sin^2\theta_b}$$

Fore-arc

$$\psi = \frac{-rv\left[\left(\theta_f - \theta\right)\sin\theta + \theta\sin\left(\theta_f - \theta\right)\right]}{\theta_f + \sin\theta_f}$$

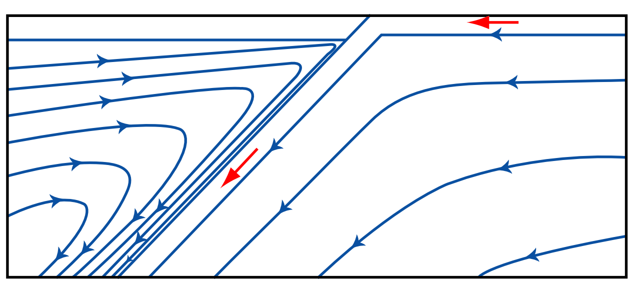


Figure 26.3 Stream lines for flow within the mantle. Motion is with respect to the plate behind the island arc and is driven by the motion of the other plate and of the sinking slab. Thermal convection outside the slab is neglected, and the lithosphere is 50 km thick.

Figure by MIT OCW.

Stresses

$$\frac{\partial v_r}{\partial r} = 0,\quad \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_r}{r} = 0 \;\Rightarrow \tau = \begin{pmatrix} -p & \tau_{r\theta} \\ \tau_{r\theta} & -p \end{pmatrix}$$

Back-arc

$$\tau_{r\theta} = \frac{2v\eta}{r}\frac{\left[\theta_b\cos\left(\theta_b - \theta\right) - \sin\theta_b\cos\theta\right]}{\theta_b^{\,2} - \sin^2\theta_b}$$

Fore-arc

$$\tau_{r\theta} = \frac{2v\eta}{r}\frac{\cos\theta + \cos\left(\theta_f - \theta\right)}{\theta_f + \sin\theta_f}$$

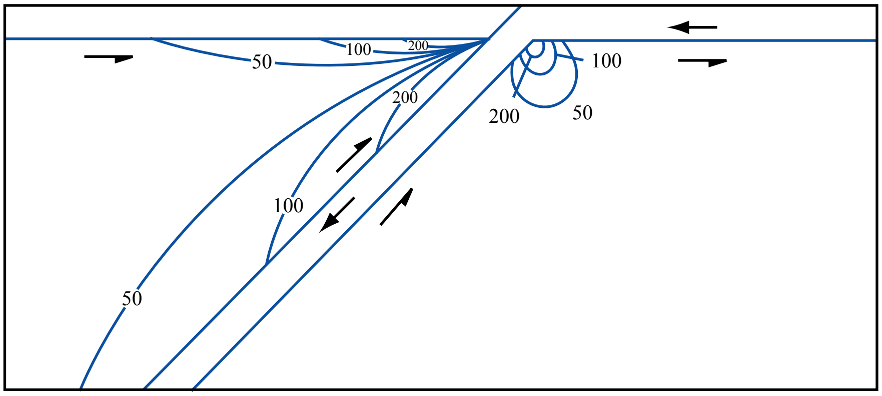


Figure 26.4 Shear stresses in bars caused by the flow in figure 26.3

Figure by MIT OCW.

Pressure – no direct equation

But – equilibrium $\tau_{ij,j} = 0$

$$-\frac{\partial p}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} = 0$$

$$-\frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{2}{r}\tau_{r\theta} + \frac{\partial \tau_{r\theta}}{\partial r} = 0$$

Back-arc

$$p = -\frac{2\nu\eta}{r}\frac{\left[\theta_b \sin\left(\theta_b - \theta\right) + \sin\theta_b \sin\theta\right]}{\theta_b^{\,2} - \sin^2\theta_b} < 0$$

$$\rightarrow -\frac{2\nu\eta}{r}\frac{\theta_b \sin\theta_b}{\theta_b^{\,2} - \sin^2\theta_b} \text{ at surface} \qquad \rightarrow -\frac{2\nu\eta}{r}\frac{\sin^2\theta_b}{\theta_b^{\,2} - \sin^2\theta_b} \text{ on slab}$$

Fore-arc

$$p = \frac{2\nu\eta}{r}\frac{\sin\theta - \sin\left(\theta - \theta_f\right)}{\theta_f + \sin\theta_f} > 0$$

$$\rightarrow \frac{2\nu\eta}{r}\frac{\sin\theta_f}{\theta_f + \sin\theta_f} \text{ at surface and on slab}$$

$\Rightarrow$ excess pressure in fore-arc
suction in back-arc – tendency to lift slab (and also modify surface topography)

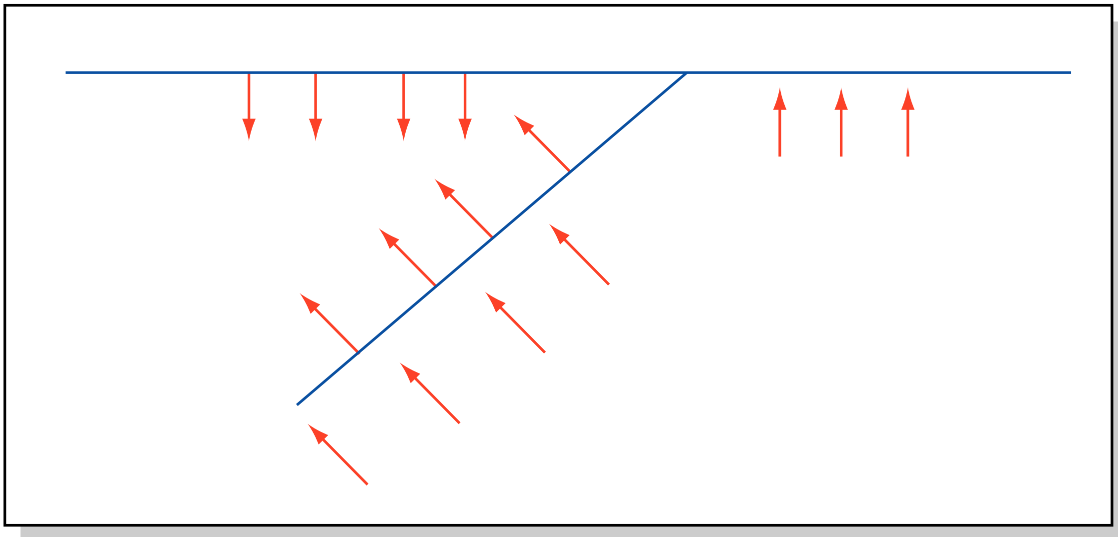


Figure by MIT OCW.

This flow tends to lift slab –

Resisted by

1) gravity
2) resistance to squeezing material out of wedge [not well posed for ∞ wedge → needs ∞ work]

Consider gravity – torque balance

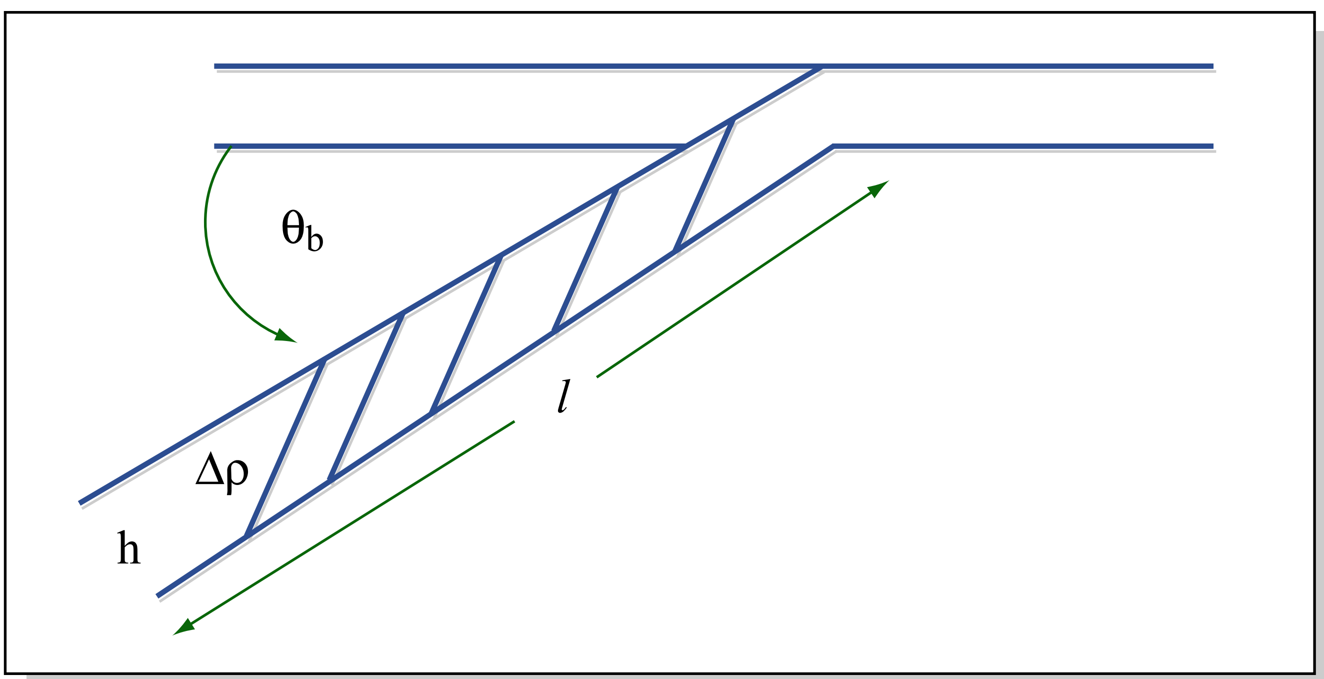


Figure by MIT OCW.

$$T_{flow} = \int_0^l \left(P_f - P_b\right)\cdot r dr$$

$$= 2\eta v l\left[\frac{\sin\theta_b}{\left(\pi-\theta_b\right)+\sin\theta_b}+\frac{\sin^2\theta_b}{\theta_b{}^2-\sin^2\theta_b}\right]$$

$$T_{grav} = \int_0^l \Delta\rho g h r\cos\theta_b dr$$

$$= \frac{1}{2}\Delta\rho g h l^2\cos\theta_b$$

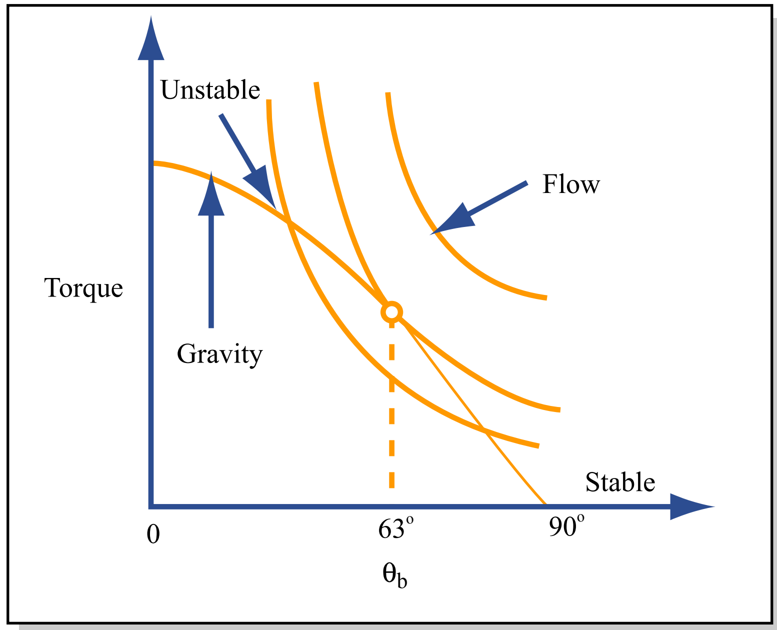
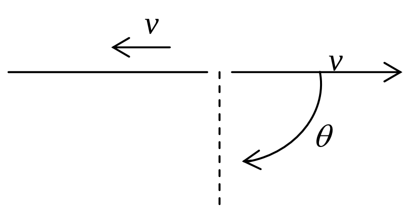


Figure by MIT OCW.

Ridge problem



By symmetry, at $\theta = \frac{\pi}{2}$, $v_\theta = 0$, $\tau_{r\theta} = 0$

Let $\psi = r\Theta$

$$v_r = \Theta' \qquad \tau_{r\theta} = \frac{\eta}{r}\left(\Theta''+\Theta\right)$$

$$v_\theta = -\Theta \qquad p = -\frac{\eta}{r}\left(\Theta'''+\Theta'\right)$$

$$\nabla^4\psi = 0$$

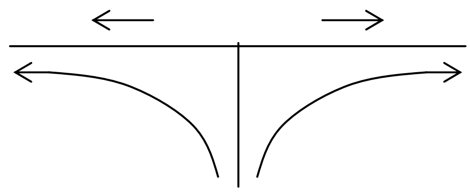
$$\Theta = A\sin\theta + B\cos\theta + C\theta\sin\theta + D\theta\cos\theta$$

$\theta = 0$	$v_\theta = 0 \Rightarrow B = 0$ $v_r = v = A + D$ $A = v,\ C = -(2/\pi)v$	$\theta = \pi/2$	$v_\theta = 0 \Rightarrow A + (\pi/2)C = 0$ $\tau_{r\theta} = 0 \Rightarrow D = 0$

$$v_r = \frac{2v}{\pi}\left[\left(\frac{\pi}{2}-\theta\right)\cos\theta - \sin\theta\right] \qquad \tau_{r\theta} = \frac{4\eta v}{\pi r}\cos\theta$$

$$v_\theta = \frac{-2v}{\pi}\left(\frac{\pi}{2}-\theta\right)\sin\theta \qquad p = -\frac{4\eta v\sin\theta}{\pi r}$$

Note: at $\theta = \frac{\pi}{2}$, $v_r = -\frac{2}{\pi} v$ (related to gradient in stream function)



$\Rightarrow$ upwelling zone wider than horizontal flow zone

$\Rightarrow$ zone of magma generation wider than depth from which melt segregates