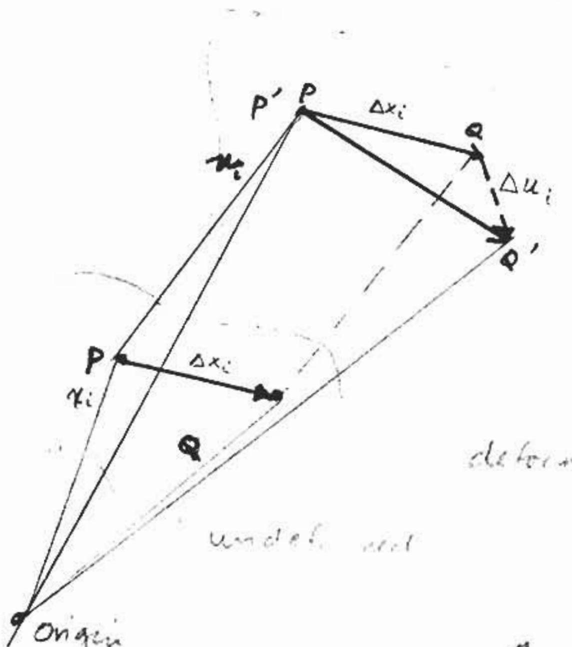
Infinitesimal Strain (2 of 8)



Point P carried to P'

$[O\underset{\sim}{P}] = x_i$

$[OP' - OP] = u_i$ (translation)

Consider vector PQ.

$[\overline{PQ}] = \Delta x_i$

Point Q carried to Q'

$OQ' = x_i + \Delta x_i + u_i + \Delta u_i$

Are not interested in translation part

$$\Delta PQ = \overline{QQ'} = P'Q' - PQ = \Delta u_i$$

but for any continuous diff. variable

$$\Delta u_i = \frac{\partial u_i}{\partial x_j} \Delta x_j$$

$$\Delta u_i = e_{ij} \Delta x_j$$

gradient of change for an arbitrary displacement.

Define $e_{ij} = \dfrac{\partial u_i}{\partial x_j}$ Displacement Gradient Tensor

Strain

Statements

1.) e_{ij} is a second rank tensor

$$u_i' = a_{ik} u_k$$ (components of a vector transform like this)

$$x_l = a_{jl} x_j'$$ (old displacement i.t.o. new)

$$\Rightarrow \frac{\partial}{\partial x_j'} = \frac{\partial}{\partial x_l}\frac{\partial x_l}{\partial x_j'} = a_{jl}\frac{\partial}{\partial x_l}$$

then $$\frac{\partial}{\partial x_j'} u_i' = a_{ik} a_{jl} \frac{\partial u_k}{\partial x_l}$$

$$\Rightarrow \boxed{e'_{ij} = a_{ik} a_{jl} e_{kl}}$$

2.)

Divide tensor into 2 parts:

$$\tfrac{1}{2}(e_{ij} + e_{ji}) \triangleq \epsilon_{ij}$$ { infinitesimal strain tensor sym.

$$\tfrac{1}{2}(e_{ij} - e_{ji}) \triangleq \omega_{ij}$$ { rotation tensor antisym.

e_{ij}

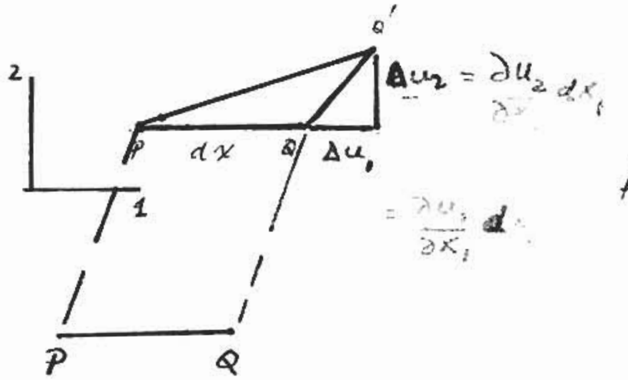
$$\boxed{e_{ij} = \epsilon_{ij} + \omega_{ij}}$$

any tensor decomposed into sym. & anti sym.

Specific Examples of Strain

Now consider a special case

$PQ = \hat{e}_1 dx$ where $\hat{e}_1$ unit vector along x_1 axis



$$\Delta u_i = \frac{\partial u_i}{\partial x_j} \Delta x_j \quad \text{for any diff. fn.}$$

for example

$$\Delta u_1 = \frac{\partial u_1}{\partial x_1} \Delta x_1 + \frac{\partial u_1}{\partial x_2} \Delta x_2 + \frac{\partial u_1}{\partial x_3} \Delta x_3$$

by definition $\Delta u_i = e_{11} \Delta x_1 + e_{12} \Delta x_2 + e_{13} \Delta x_3$

$\therefore e_{11}$ is change of length of $dx_1 \vec{i}$ in 1 direction
<u>normal strain</u> note: $\frac{l + \Delta l}{l}$ = stretch = $1 + e_{11}$

e_{12} is change of length of $dx_i \vec{i}$ in 2 direction
shear strain

note: $\frac{\partial u_2}{\partial x_1} dx_1 \Big/ dx_1 + \frac{\partial u_1}{\partial x_1} dx_1 = \tan\theta$

but $\theta \ll 1$ so $\tan\theta \simeq \theta$

and $\partial u_2 / \partial x_1 \ll 1$ so $dx_1 + \frac{\partial u_1}{\partial x_1} dx_1 \simeq dx_1$

$$\Rightarrow \frac{\partial u_2}{\partial x_1} = e_{21} \simeq \theta$$

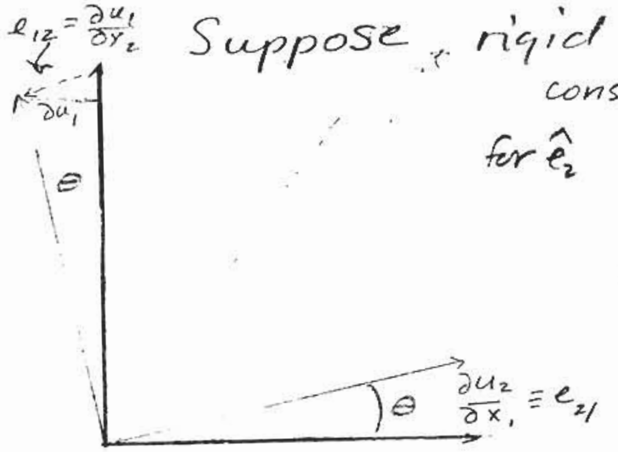
so e_{11} measure change of length in 1 direction of a line in 1 direction

e_{12} measures change of length in 1 dir of a line (originally) in 2 direction

Infinitesimal Strain

Example 2

Suppose rigid body rotation



consider rotation of vectors $\hat{e}_1$ and $\hat{e}_2$

for $\hat{e}_2$ $\quad \frac{\partial u_1}{\partial x_2} = -\theta \quad$ because $\partial u_1 < 0$

from definition

$$\omega_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right)$$

$$= \frac{1}{2}(-\theta - +\theta) = -\theta$$

but $\epsilon_{12} = \frac{1}{2}(e_{12} + e_{21}) = 0$

Particular Special Types of Strain

Homogenous strain:

All elements in body strained same

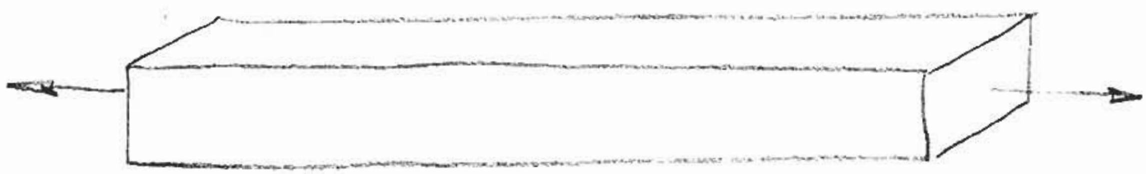
$$\epsilon_{ij} \neq \epsilon_{ij}(x_1, x_2, x_3)$$

st lines → st lines
p// line → p// lines
all st. lines
ext. cont.
by same ratio

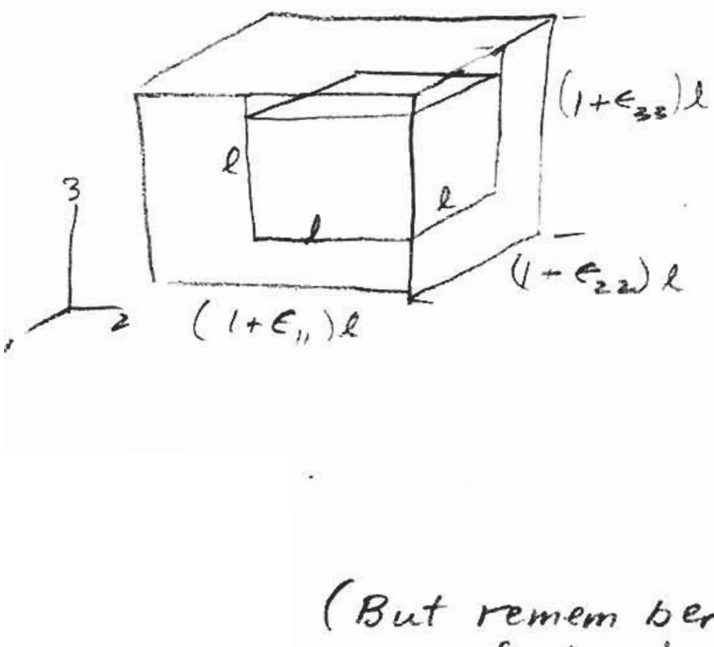
an ellipse becomes a diff. ellipse
" circle " "n ellipse

Heterogenous Strain: $\quad \epsilon_{ij} = \epsilon_{ij}(x_k)$

Plane Strain All elements in a plane in original remain in a plane in deformed body (actually impossible) but nearly achieved if z(3) dimension very large compared to xy(2) dimensions



Volumetric Strain



$$V = (1+\epsilon_{33})(1+\epsilon_{22})(1+\epsilon_{11})\ell^3$$

$$= \ell^3 (1 + \epsilon_{11} + \epsilon_{22} + \epsilon_{33} + \dots)$$

fractional change in vol

$$\frac{\Delta V}{V} = \frac{\ell^3(1+\epsilon_{11}+\epsilon_{22}+\epsilon_{33}) - \ell^3}{\ell^3}$$

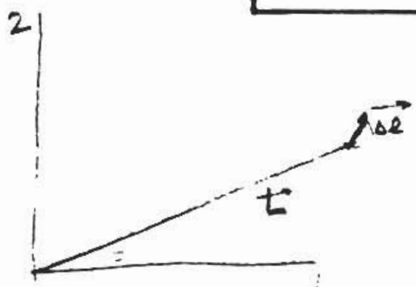
$$= \epsilon_{11} + \epsilon_{22} + \epsilon_{33}$$

(But remember trace of tensor is one of the invarients of tensor.)

Summary: Strain

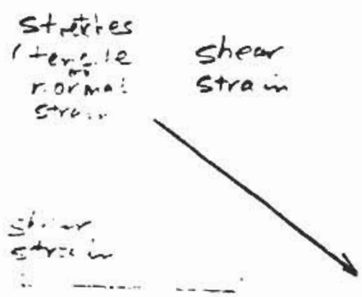
- Strain is the symmetric part of the deformation gradient tensor

$$\boxed{\begin{aligned}\epsilon_{ij} &\triangleq \tfrac{1}{2}\left[e_{ij} + e_{ji}\right] \\ &= \tfrac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]\end{aligned}}$$



$$\underset{\sim}{\Delta l} = \underset{\approx}{\epsilon}\ \underset{\sim}{L}$$

$$\Delta L_i = \epsilon_{ij} L_j \quad \left(= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) dx_j \right)$$



- Mohr's circle construction applies for 2 dimensional strain

  Also because symmetric 2nd rank tensor

  $\Rightarrow$ 3 principal directions

  3 principal values.

- stretch in an arbitrary direction

$$a_j = \vec{L} \cdot \vec{e}_j = \ldots$$

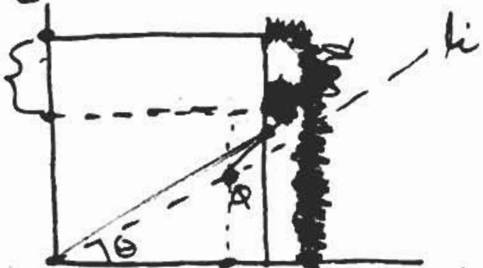
$$\Delta L_i = \epsilon_{ij} (\vec{L} \cdot \vec{e}_j) = \epsilon_{ij}\, a_j |L|$$

$$\lambda = \frac{\Delta L_i}{|L|} = \epsilon_{ij}\, a_j \quad \text{where } a_j \text{ is } \cos(\angle \text{ between } \vec{e}_j \text{ and } \vec{L})$$

I B 3 c 2

Example :

$\Delta l_2 = \epsilon_{22} l_2$



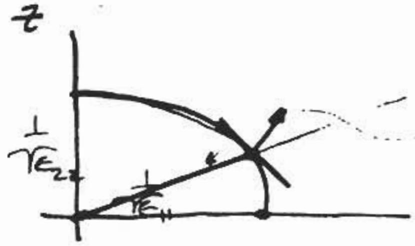
Body

$\Delta l_1 = \epsilon_{11} l_1$

No shear along axes $\Rightarrow$ these must be principal axes

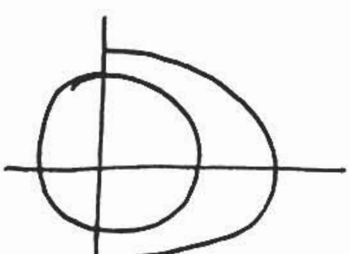
Assume $\epsilon_{11} < \epsilon_{22}$

$$\frac{1}{\sqrt{\epsilon_{11}}} > \frac{1}{\sqrt{\epsilon_{22}}}$$



Quadric

$$\Delta l_i = \epsilon_{ij} l_j$$



Ellipsoid

Equation of circle = $x^2 + y^2 = 1$

Equation of deformed ellipse

$$x \rightarrow (1+\epsilon_1) x = x'$$

$$\frac{x'}{1+\epsilon_1} = x$$

$$\Rightarrow \frac{x'^2}{(1+\epsilon_1)^2} + \frac{y'^2}{(1+\epsilon_2)^2} + \frac{z'^2}{(1+\epsilon_3)^2} = 1$$

__Homework:__

Exercise 6.1 in Nye: A small def. is defined by the tensor

$$e_{ij} = \begin{bmatrix} 8 & -1 & -1 \\ 1 & 6 & 0 \\ -5 & 0 & 2 \end{bmatrix} \times 10^{-6}$$

Find magnitudes of principal