PV fronts: dispersion relations

Consider a line

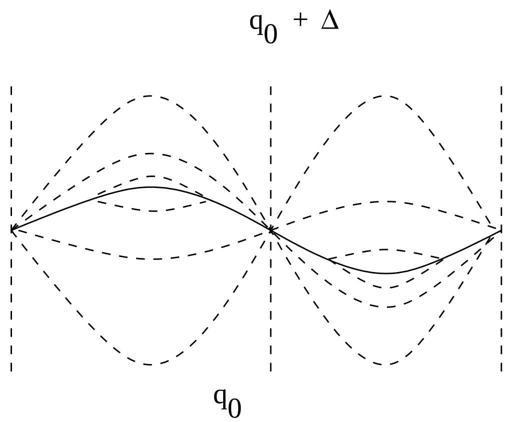
$$Y = \eta(x,t)$$

separating two regions of constant PV. South of the front, we have

$$q = q_0$$

while to the north

$$q = q_0 + \Delta$$



Front separating two constant PV regions. The dashed lines show the anomaly in ψ.

The interface between the two regions is a material surface; fluid parcels cannot move from one side to the other, since they would have to change their PV which is prohibited. Therefore, we have

$$\boxed{\frac{\partial}{\partial t}\eta + u(x,\eta,t)\frac{\partial}{\partial x}\eta = v(x,\eta.t)}$$

The PV inversion formula gives

$$\boxed{\nabla^2\psi - \gamma^2\psi = \begin{cases} q_0 + \Delta & y > \eta \\ q_0 & y < \eta \end{cases}}$$

with $\gamma = 0$ (BT) or $1/R_d$ (BC). We'll discuss two ways of solving the linearized problem where

$$\eta = \eta_0 \exp(\imath kx)$$

Matching

We can solve the PV equation using a Fourier expansion in x; for the linearized problem we only need the x-independent and $\exp(\imath kx)$ modes. We'll talk about the barotropic case:

$$\psi \simeq \begin{cases} (q_0+\Delta)\dfrac{y^2}{2} + A\exp(\imath kx - ky) & y > \eta \\ q_0\dfrac{y^2}{2} + B\exp(\imath kx + ky) & y < \eta \end{cases}$$

Now we match ψ (or v) and $\frac{\partial}{\partial y}\psi$ (or u) at the perturbed boundary keeping terms of order η, A, B but not higher order. Matching ψ gives

$$A\exp(\imath kx) = B\exp(\imath kx) \quad \Rightarrow \quad A = B$$

Matching $\frac{\partial}{\partial y}\psi$ gives

$$(q_0+\Delta)\eta_0\exp(\imath kx) - kA\exp(\imath kx) = q_0\eta_0\exp(\imath kx) + kA\exp(\imath kx) \quad \Rightarrow \quad A = \frac{1}{2k}\Delta\eta_0$$

The linearized kinematic condition for the interface tells us we need u to order one and v to order η. These are just 0 and

$$v(x,\eta,t) \simeq \frac{\Delta}{2k}\frac{\partial}{\partial x}\eta$$

respectively. Therefore, the interface evolves according to

$$\frac{\partial}{\partial t}\eta = \frac{\Delta}{2k}\frac{\partial}{\partial x}\eta$$

or

$$\omega = -\frac{\Delta}{2} \quad , \quad c = -\frac{\Delta}{2k}$$

Greens function

Alternatively, we split ψ into a background and a fluctuation part

$$(\nabla^2 - \gamma^2)\overline{\psi} = q_0 + \Delta\mathcal{H}(y)$$
$$(\nabla^2 - \gamma^2)\psi' = \Delta[\mathcal{H}(y-\eta) + \mathcal{H}(y)]$$

where $\mathcal{H}$ is the Heaviside step function. Differentiating the first equation by y gives

$$(\frac{\partial^2}{\partial y^2} - \gamma^2)\overline{u} = -\Delta\delta(y)$$

Linearizing the second by Taylor-expanding $\mathcal{H}(y-\eta) \simeq \mathcal{H}(y) - \eta\delta(y)$... gives

$$(\nabla^2 - \gamma^2)\psi' = -\Delta\eta\delta(y)$$

For a single wave, the last equation becomes

$$(\frac{\partial^2}{\partial y^2} - k^2 - \gamma^2)\psi' = -\Delta\eta\delta(y)$$

Defining the Greens function

$$(\frac{\partial^2}{\partial y^2} - \alpha^2)G_\alpha(y-y') = \delta(y-y')$$

gives

$$\overline{u} = -\Delta G_\gamma(y)$$
$$\psi' = -\Delta\eta\ G_K(y)$$

with $K = \sqrt{k^2+\gamma^2}$ and our interface satisfies

$$\frac{\partial}{\partial t}\eta = \Delta G_\gamma(0)\frac{\partial}{\partial x}\eta - \Delta G_K(0)\frac{\partial}{\partial x}\eta$$

so that

$$c = \Delta G_K(0) - \Delta G_\gamma(0)$$

For the barotropic problem

$$G_k(0) = -\frac{1}{2k} \quad , \quad G_0(0) = 0 \quad \Rightarrow \quad c = -\frac{\Delta}{2k}$$