

14.122 Problem Set #5

1. Suppose two firms are engaged in repeated Cournot competition with demand being  $P(q_1, q_2) = 1 - (q_1 + q_2)$ . Show that using punishments which involve a permanent reversion to the static Cournot equilibrium ( $q_1 = q_2 = \frac{1}{3}$ ) the firms can support full collusion ( $q_1 = q_2 = \frac{1}{4}$ ) as a subgame perfect equilibrium provided that  $\delta \geq \frac{9}{17}$ .

2. A payoff vector  $v = (v_1, v_2, \dots, v_I) \in \mathbb{R}^I$  is said to be *feasible* in a game  $G$  if there exist action profiles  $a^1, a^2, \dots, a^k$  and nonnegative weights  $w^1, w^2, \dots, w^k$  with  $\sum_j w^j = 1$  such that

$$v_i = w^1 g_i(a^1) + w^2 g_i(a^2) + \dots + w^k g_i(a^k).$$

A payoff vector  $v$  is said to be *strictly individually rational* if

$$v_i > \min_{\sigma_{-i} \in \Sigma_{-i}} \max_{\sigma_i \in \Sigma_i} g_i(\sigma_i, \sigma_{-i})$$

for all  $i$ .

(a) What are the feasible and individually rational payoffs in the Cournot game.

(b) In answering (a) you should have noted that the feasible individually rational payoff vectors for the Cournot game include some where the players receive *lower* payoffs than they would if they repeatedly played the static Cournot equilibrium. Use the no single period deviation criterion to show that the symmetric strategy profile  $(q, q)$  described below is a subgame perfect equilibrium if  $\delta \geq \frac{2}{3}$  and show that the “average” payoffs players receive from it corresponds to a payoff vector in which each player earns less than the Cournot profit.

$$q(h_t) = \begin{cases} 3/8 & \text{if } t = 0 \\ 3/8 & \text{if } t > 0 \text{ and both players followed strategy } q \text{ in the previous period} \\ 1/2 & \text{otherwise} \end{cases}$$

(Be sure to note that the “punishment”  $q_1 = q_2 = \frac{1}{2}$  lasts for only one period after a deviation.)

Comment on how the punishments above are similar to and different from those which I showed in class make it possible for firm 1 to produce  $q = 1$  in a Nash equilibrium of the Stackelberg model.

For extra credit you could try to describe how one might construct punishments related to those above (but different) to sustain full collusion in the repeated Cournot game for some discount factors which are less than  $\frac{9}{17}$ .

3. Two players are working together to try to complete a project. When players 1 and 2 choose effort levels  $e_1, e_2 \in [0, 1]$ , the probability that the project is successfully completed is  $\frac{1}{2}(1 + e_1)e_2$ . Assume that players receive one util if the project is successful, and that they incur a disutility of effort which makes their expected payoff in the game equal to

$$u_i(e_1, e_2; c_1, c_2) = \frac{1}{2}(1 + e_1)e_2 - c_i e_i^2,$$

where  $c_1$  and  $c_2$  are constants greater than or equal to one which parameterize the disutility of effort.

Assume that it is common knowledge that  $c_2$  is equal to one, but that  $c_1$  is known only to player 1, with player 2's prior being that nature chooses  $c_1$  from the probability density function

$$f(x) = \begin{cases} \frac{2}{5}x & \text{if } x \in [2, 3] \\ 0 & \text{otherwise.} \end{cases}$$

Find the Bayesian Nash equilibrium of this game.

4. Consider the following strategic situation. Two opposed armies are poised to seize an island which is currently unoccupied. Each army's general can choose either to "attack" or to "not attack". In addition, each army is either strong or weak with its strength being known to its general alone. General 1's prior is that army 2 is strong with probability  $\frac{1}{2}$  (and that this event is independent of army 1's strength) and general 2's prior is that army 1 is strong with independent probability  $\frac{1}{3}$ . These priors are common knowledge among the players.

Suppose that the payoffs in the game are reflect additively both the benefit of capturing the island and cost suffered in fighting. Specifically assume that the island is worth  $M$  if captured, and that an army captures the island if and only if it attacks when its opponent does not or it attacks when its rival does also, but it is strong and its rival is weak. (Note that when two weak or two strong armies fight neither captures the island.) The cost of fighting is  $s$  if an army is strong and  $w$  if an army is weak (with  $s < w$ ) regardless of the strength of the attacking rival army (with no fighting costs being incurred if one or both armies choose not to attack and there is no battle).

Identify all of the pure strategy Bayesian Nash Equilibria of this game.

5. Glenn Ellison and Josh Angrist are trying to meet for lunch. At twelve o'clock each of them must go to one of two places: Florentina's or the trucks behind the swimming pool. Because they need to talk about something, assume that each receives a benefit of  $\frac{2}{3}$  utils if they meet. Assume also that each incurs a disutility cost of  $w$  from waiting in line outside Florentina's in the cold. Because Josh came in on the subway just before noon he knows how long the Florentina's line is, i.e. he knows  $w$ , while Glenn's prior is that  $w$  is distributed uniformly on  $[0, 1]$ . While it is common knowledge that Josh is completely indifferent between the food choices at each place, assume that Glenn incurs a disutility of  $3d$  if he eats at the trucks. The strength of Glenn's dislike for the trucks varies from day to day — assume that Glenn knows  $d$  while Josh's prior is that  $d$  is distributed uniformly in  $[0, 1]$ .

To summarize the payoffs, when the wait at Florentina's is  $w$  and Glenn's disutility from going to the trucks is  $d$  the payoffs in the game are (with Josh as player 1)

	Florentina	Trucks
Florentina	$\frac{2}{3} - w, \frac{2}{3} - w$	$-w, -3d$
Trucks	$0, -w$	$\frac{2}{3}, \frac{2}{3} - 3d$

(a) How would you specify this game formally as a Bayesian game? In particular, what would the sets of types be and how would the payoff functions depend on the types? What form would you expect the equilibrium strategies to take?

(b) Find the Bayesian Nash equilibrium of this game. How often do Glenn and Josh have lunch together?