Recursive Methods

Outline Today's Lecture

- finish Euler Equations and Transversality Condition
- Principle of Optimality: Bellman's Equation
- Study of Bellman equation with bounded F
- contraction mapping and theorem of the maximum

Infinite Horizon $T=\infty$

$$V^{*}\left(x_{0}\right)=\sup_{\left\{x_{t+1}\right\}_{t=0}^{\infty}}\sum_{t=0}^{\infty}\beta^{t}F\left(x_{t},x_{t+1}\right)$$

subject to,

$$x_{t+1}\in\Gamma\left(x_{t}\right) \tag{1}$$

with x_0 given

- $\sup\{\}$ instead of $\max\{\}$
- define $\left\{x'_{t+1}\right\}_{t=0}^{\infty}$ as a plan
- define $\Pi\left(x_{0}\right)\equiv\left\{\left\{x'_{t+1}\right\}_{t=0}^{\infty}|x'_{t+1}\in\Gamma\left(x'_{t}\right)\text{ and }x'_{0}=x_{0}\right\}$

Assumptions

A1. $\Gamma(x)$ is non-empty for all $x \in X$
A2. $\lim_{T\to\infty} \sum_{t=0}^{T} \beta^t F(x_t, x_{t+1})$ exists for all $x \in \Pi(x_0)$
then problem is well defined

Recursive Formulation: Bellman Equation

- value function satisfies

$$
\begin{aligned}
V^*(x_0) &= \max_{\substack{\{x_{t+1}\}_{t=0}^{\infty} \\ x_{t+1}\in\Gamma(x_t)}} \left\{ \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \right\} \\
&= \max_{x_1\in\Gamma(x_0)} \left\{ F(x_0, x_1) + \max_{\substack{\{x_{t+1}\}_{t=1}^{\infty} \\ x_{t+1}\in\Gamma(x_t)}} \sum_{t=1}^{\infty} \beta^t F(x_t, x_{t+1}) \right\} \\
&= \max_{x_1\in\Gamma(x_0)} \left\{ F(x_0, x_1) + \beta \max_{\substack{\{x_{t+1}\}_{t=1}^{\infty} \\ x_{t+1}\in\Gamma(x_t)}} \sum_{t=0}^{\infty} \beta^t F(x_{t+1}, x_{t+2}) \right\} \\
&= \max_{x_1\in\Gamma(x_0)} \left\{ F(x_0, x_1) + \beta V^*(x_1) \right\}
\end{aligned}
$$

continued...

- idea: use BE to find value function V^* and policy function g [Principle of Optimality]

Bellman Equation: Principle of Optimality

- Principle of Optimality idea: use the functional equation

$$V(x) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta V(y)\}$$

  to find V^* and g

- note: nuisance subscripts t, $t+1$, dropped
- a solution is a function $V(\cdot)$ the same on both sides
- **IF** BE has unique solution then $V^* = V$
- more generally the "right solution" to (BE) delivers V^*