

## 14.27 Problem Set 4

Due 11/12

### Question 3: Applying the Revenue Equivalence Theorem

In class we discussed the Revenue Equivalence Theorem, which showed that under a set of standard assumptions many different auction mechanisms raise precisely the same expected revenue for the seller. An additional consequence of the Revenue Equivalence Theorem that we did not discuss is that it also guarantees that in equilibrium, each of these auction mechanisms give the same expected surplus to each bidder, conditional on their valuation  $v$ . This is also known as “Payoff Equivalence.” In particular, for a bidder with valuation  $v$ , the expected utility of this bidder in equilibrium is:

$$EU(v) = \int_v^v F(t)^{(n-1)} dt \quad (1)$$

The purpose of this question is to demonstrate how this can be very useful as a short cut to determining the equilibrium bidding strategies of auction mechanisms that are more complex than our benchmark case, the second price auction, which of course is simply  $b^*(v_i) = v_i$ .

a)

Write down the equation for the bidder’s expected utility as a function of  $v$ ,  $b(v)$ , and  $F(v)$  for the first price auction.

$$E[U(v_i)] = (v_i - b(v_i)) Pr(b(v_i) > b(v_j) \forall j \neq i)$$

Assuming  $b(\cdot)$  is monotone increasing,  $Pr(b(v_i) > b(v_j) \forall j \neq i) = Pr(v_i > v_j \forall j \neq i)$ . Since values are independent,  $Pr(v_i > v_j \forall j \neq i) = Pr(v_i > v_j)^{N-1} = F(v_i)^{N-1}$ . Then:

$$E[U(v_i)] = (v_i - b(v_i)) F(v_i)^{N-1}$$

b)

Use this “Payoff Equivalence” result explained above (Equation 1) combined with your equation in (a) to derive the general expression for the equilibrium bidding function for a bidder in a first price auction. (This would have been very difficult for the first price auction without this trick.)

We have that:

$$(v_i - b(v_i)) F(v_i)^{N-1} = \int_v^v F(t)^{(n-1)} dt$$

$$\Rightarrow b^*(v_i) = v_i - \frac{\int_v^v F(t)^{(n-1)} dt}{F(v_i)^{N-1}}$$

c)

Now let  $F()$  be specified as  $U[0, 1]$ . Plug this into part (b) to derive the equation that the Professor gave in class for the first price auction equilibrium bidding strategy.

$$b^*(v_i) = v_i - \frac{\int_{\underline{v}}^v t^{n-1} dt}{v_i^{N-1}} = v_i - \frac{v_i^N}{v_i^{N-1}} = \frac{N-1}{N} \cdot v_i$$

Now consider a “Sealed Bid All Pay Auction,” where each of the  $N$  bidders submits their bid, the highest bid wins the object, but every single bidder must pay their submitted bid even if they don’t win the object.

d)

Write down the equation for the bidder’s expected utility as a function of  $v$ ,  $b(v)$ , and  $F(v)$  for the all pay auction.

$$\begin{aligned} E[U(v_i)] &= (v_i - b(v_i)) Pr(b(v_i) > b(v_j) \forall j \neq i) - b(v_i) [1 - Pr(b(v_i) > b(v_j) \forall j \neq i)] \\ &= (v_i - b(v_i)) F(v_i)^{N-1} - b(v_i) [1 - F(v_i)^{N-1}] \\ &= v_i F(v_i)^{N-1} - b(v_i) \end{aligned}$$

e)

Use this “Payoff Equivalence” result explained above (Equation 1) combined with your equation in (d) to derive the general expression for the equilibrium bidding function for a bidder in the all pay auction.

$$\begin{aligned} v_i F(v_i)^{N-1} - b^*(v_i) &= \int_{\underline{v}}^v F(t)^{(n-1)} dt \\ b^*(v_i) &= v_i F(v_i)^{N-1} - \int_{\underline{v}}^v F(t)^{(n-1)} dt \end{aligned}$$

f)

Now let  $F()$  be specified as  $U[0, 1]$ . Calculate the expected revenue to the seller in the all pay auction.

$$\begin{aligned} b^*(v_i) &= \frac{N-1}{N} \cdot v_i^N \\ \text{Expected revenue: } NE[b^*(v)] &= (N-1) E[v^N] \\ &= (N-1) \int_0^1 v^N dv = \frac{N-1}{N+1}. \end{aligned}$$

g)

Despite these revenue and payoff equivalence results, why do you think that the all pay auction is almost never used in real life?

Fun note: the all pay auction serves as a very nice model of political lobbying / bribery since interest groups on all sides pay their sunk costs in lobbying fees and yet only one decision can ultimately be made on an issue.

One explanation for why we almost never see this auction in practice is risk aversion. If bidders are risk averse, they will prefer an auction where they do not pay unless they win to an all pay auction. This will imply that in an all pay auction risk averse bidders will bid less, and revenue will be lower.

## Question 4: Second Degree Price Discrimination

A monopolist is studying different pricing schemes in a market. It knows that there are two types of consumers in the market, 1 and 2, in equal proportion. Consumer type 1 is willing to pay \$20 for a high quality unit and \$10 for a low quality unit, while consumer type 2 is willing to pay \$13 for a high quality unit and \$9 for a low quality unit. Assume that the marginal cost of production is 0.

a)

If the monopolist offers a single product, what quality level does it choose and what price does it charge?

- Alternative 1: Sell high quality product to all consumers.
  - If selling to all consumers,  $P = 13$ . A lower price results in the same quantity sold, and lower profits.
  - $\pi = 2N \cdot 13 = 26N$ .
- Alternative 2: Sell high quality product to consumer type 1 only.
  - $P = 20$ ,  $\pi = 20N$ .
- Alternative 3: Sell low quality product to all consumers.
  - $P = 9$ ,  $\pi = 2N \cdot 9 = 18N$
- Alternative 4: Sell low quality product to all consumers. Clearly suboptimal.
- If the monopolist only offers one product, it offers the high quality product at a price of 13. Notice that if there was a marginal cost of production, and producing the high quality good is more expensive than the low quality good, this result could easily have changed to the monopolist selling only to consumer type 1 or to selling the low quality good to both.

b)

If the monopolist implements a second degree price discrimination scheme by offering both quality levels at different prices  $((p_H, q_H), (p_L, q_L))$ , what prices and quality levels does it set?

The monopolist solves:

$$\max_{p_H, p_L} Np_H + Np_L$$

s.t.

$$\text{IR1: } 20 - p_H \geq 0$$

$$\text{IR2: } 9 - p_L \geq 0$$

$$\text{IC1: } 20 - p_H \geq 10 - p_L$$

$$\text{IC2: } 9 - p_L \geq 13 - p_H$$

Notice that from IC1:

$$10 + p_L \geq p_H$$

And IR2 requires that

$$9 \geq p_L$$

Therefore, the highest  $p_L$  can be is 9. If  $p_L = 9$ , then IC1 is  $19 \geq p_H$ , and if IC1 holds then IR1 must hold too. If  $p_L < 9$ , IC1 is even stricter, and again if IC1 holds then IR1 must hold too. Therefore, we can ignore IR1, as the combination of IC1 and IR2 implies that IR1 is satisfied.

Furthermore, let us ignore IC2, and check at the end whether it is met.

Then we have:

$$\text{IR2: } 9 \geq p_L$$

$$\text{IC1: } 10 + p_L \geq p_H$$

Notice that increasing  $p_L$  implies that the firm gains more profits from type 2 customers, and that it also can charge a higher  $p_H$ . Therefore,  $p_L = 9$ . This is the highest price the firm can charge low types and still get them to buy. Plugging into IC1,  $p_H = 19$ .

Let's check IC2:  $9 - p_L \geq 13 - p_H \Leftrightarrow p_H - p_L = 10 \geq 4$ . It is met.

c)

Compare profits and consumer surplus under both settings. What scheme is socially optimal? Give intuition for your answer.

Under a single product monopolist,  $\pi = 26N$ ,  $CS = (20 - 13)N = 7N$ , and welfare is  $33N$ .

Under second degree price discrimination,  $\pi = 19N + 9N = 28N$ ,  $CS = (20 - 19)N = N$ , and welfare is  $29N$ .

Welfare is lower under second degree price discrimination because type 2 consumers now receive a lower quality good, which they value less, and the marginal cost of producing that good is identical to the cost of producing the high quality good. The monopolist uses the low quality good as a way to screen consumers, allowing them to set a higher price to the high valuation consumers and gain greater profits. Note that welfare from high valuation consumers is the same in both settings, as the higher price is just a transfer from consumers to producers.

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