

# Problem set 1. 14.461 Fall 2012.

Ivan Werning

September 13, 2012

## *References:*

1. Ljungqvist L., and Thomas J. Sargent (2000), "Recursive Macroeconomic Theory," – *sections 17.2* for Problem 1,2.
2. Werning Ivan (2007), "Optimal Fiscal Policy with Redistrtribution," – *pp. 953-954* for Problem 3.
3. Chari V.V. and Patrick J. Kehoe (1999), "Optimal Fiscal and Monetary Policy," – *Section 3* for Problem 5.

## **1 Taxing Luck.**

*In this problem we see that with incomplete markets capital taxation is positive in the steady state. The intuition is that the only source of consumption smoothing is capital investment and agents in the absense regulation overaccumulate capital. That is why positive tax improves wealth as it reduces interest rate that agents face and hence, reduces savings to the optimal level. Unlike Aiyagari (1995) who carries out the analysis on a more abstract level, in this problem we take a particular functional forms that allow for more direct derivation.*

Consider economy populated by unit continuum of infinitely lived agents. Time is discrete  $t = 0, 1, \dots$ . That is, for a generic agent at time  $t$  state  $\theta_t$  is realized and denote by  $\theta^t = (\theta_0, \dots, \theta_t)$  history of shocks for an agent.<sup>1</sup> Shocks are i.i.d. across time and agents.

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<sup>1</sup>Strictly speaking, there is a unit continuum  $I$  of agents and agent  $i \in I$  at time  $t$  observes a history of shocks  $\theta_i^t = (\theta_{0i}, \dots, \theta_{ti})$ . For conciseness we drop index  $i$  and refer to a generic agent instead.

Utility function is

$$u(c) = -\frac{1}{\gamma}e^{-\gamma c} \quad (1.1)$$

and agents discount at factor  $\beta$ .

In the beginning of each period  $t$  after observing  $\theta_t$  agent chooses how much effort to exert ( $l_t$ ) which affects his income ( $y_t$ ) according to function<sup>2</sup>

$$y_t = (1 - \tau_t)w_t l_t + \sqrt{1 - l_t} + \theta_t. \quad (1.2)$$

Then consumption  $c_t$  and asset holdings  $a_t$  are chosen subject to the sequential budget constraint

$$c_t + a_t = y_t + R_t a_{t-1} - T_t, \quad \forall \theta^t \quad (1.3)$$

and the No-Ponzi condition

$$\frac{a_t}{\prod_{i=1}^t R_i} \rightarrow_t 0 \text{ almost surely} \quad (1.4)$$

where  $R_t = 1 + (1 - \tau_t^k)(r_t - \delta)$  is after-tax gross rate of return. Policy of the government is labor tax  $\tau_t$ , capital tax  $\tau_t^k$  and lump-sum tax  $T_t$  and could not be conditioned on private history of the agent. Firms produce output employing capital and labor with production function  $F(K, L) = \sqrt{KL}$ .<sup>3</sup>

Please, answer the following questions.

1. Suppose that  $\theta_t$  is equal to 1 for all  $t$ . Define equilibrium of the deterministic model and solve Ramsey problem for this economy. Is tax on capital equal to zero?

For the rest of the questions assume that  $\theta_t$  is equal to 2 and 0 with equal probabilities.

2. Write resource constraint.
3. Consider two agents with different private histories of shocks at moment  $t$ ,  $\theta^t$  and  $\tilde{\theta}^t$ . How this affects their expectations of interest rate process  $R_t$ ? In particular, find difference  $\mathbb{E}[R_{t+i}|\theta^t] - \mathbb{E}[R_{t+i}|\tilde{\theta}^t]$  for all  $i$ . How about expectations about  $L_t$  and  $K_t$ ?

4. Solve for optimal choice of effort and define by  $y^*(\cdot)$  function of maximal income

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<sup>2</sup>In the notation instead of  $y(\theta^t)$  we write  $y_t$  when history  $\theta^t$  is implied.

<sup>3</sup>As usual define by capital letters aggregate variables.

given inputs. What are the arguments of this function, does it depend on the whole history of shocks?

5. Using Euler equation find ex ante utility of the agent as a function of initial asset holdings and interest rate process?
6. Introduce the following additional notations. Let  $A_t$  be a price of console

$$A_t = \sum_{s=0}^{\infty} (\Pi_{i=0}^t R_i)^{-1} \quad (1.5)$$

and

$$\mathbb{C}\mathbb{E}[y, \frac{\gamma}{A}] \equiv -\frac{1}{\gamma} \ln \mathbb{E} \exp[-\gamma y] \quad (1.6)$$

a certainty-equivalent for utility function  $u$ . Verify that Euler equation is satisfied by a guess

$$c_t - c_{t-1} = \Delta(A_t, R_t, w_t) + \frac{1}{A_t} (y_t^* - \mathbb{E}_{t-1} y_t^*) \quad (1.7)$$

where

$$\Delta(A, R, w) \equiv \frac{1}{\gamma} \ln[\beta R] + \frac{1}{A} (\mathbb{E}[y^*] - \mathbb{C}\mathbb{E}[y^*, \frac{\gamma}{A}]). \quad (1.8)$$

7. Suppose that planner wants to choose policy so that to maximize ex ante utility of agents.<sup>4</sup> Using ex ante utility function from (4) formulate planning problem.
8. (★) Write Lagrangian for planner's problem and demonstrate that there is a positive tax on capital in the steady state.

## 2 Computation of Equilibrium.

Consider the simplified version of the model introduced above in which  $y_t = \theta_t + w_t$  (i.e. there is no alternative home production and  $l \in [0, 1]$ ). Suppose first there is no government intervention. Implement the following algorithm to compute the equilibrium.

1. Start with some value of capital  $K = K_j$ ,  $j = 0$  and compute interest rates and wages from firms' optimality. Introduce state variable  $(\theta, a)$  and a mapping  $\alpha_j(\theta, a)$  which maps current state of a generic agent to the optimal choice of asset holdings.

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<sup>4</sup>Notice that agents are different ex post, but identical ex ante.

2. Find stationary distribution  $\lambda_j$  for  $(\theta, a)$ . Compute new capital level

$$K_j^* = \sum_{\theta, a} \lambda_j(\theta, a) \alpha_j(\theta, a). \quad (2.1)$$

3. Set  $K_{j+1} = \xi K_j + (1 - \xi) K_j^*$  for  $\xi \in (0, 1)$  and iterate.

Now compute equilibrium capital holdings for different tax levels on capital,  $\tau^k = 0.25, 0.5$   
How the tax on capital affects capital level and ex ante welfare.

### 3 Capital Levies

*This problem investigates the optimality of capital levies and their relationship to the time inconsistency problem. We consider a deterministic economy subject to linear taxation of labor and capital.*

1. Consider a representative agent economy. Assume the tax on capital is not bounded above. Show that it is always optimal to tax capital completely away, i.e. confiscate it.
2. Assume now that preferences are separable between consumption and labor and that the utility over consumption is CRRA. Assume that the tax on capital must be bounded above by  $\bar{\tau}$ . Show that the optimal initial tax on capital must be set at this bound. What happens with the optimal tax rate on capital in other periods? [Hint: you must modify the usual Ramsey problem to incorporate the constraints on capital taxation.]
3. Define a time inconsistent plan to be an optimum for the Ramsey problem at  $t = 0$  that is no longer optimal if the government could re-optimize at some date  $t > 0$  and select a new competitive equilibrium, taking the starting conditions in period  $t$  as given. Argue that your results in (a)-(b) imply a time inconsistency problem.
4. Now consider an economy with two agents. Show that a positive tax on capital may or may not be optimal. Argue that a subsidy on capital may be optimal.
5. (★) Spell out an example economy where the optimum features a constant tax on labor, no tax on capital and the plan is time consistent. Be very specific in your construction.

## 4 Budget Balance and Chamley-Judd

We now investigate whether budget-balance rules affects the long-run results on a zero capital-income tax.

1. Consider a representative agent Ramsey model with capital, but where no bonds are available. Assume the government cannot purchase capital; agents can purchase capital. Write down the “budget balance” constraint for the government and the sequential budget constraints for agents. [Note: a present value condition for the agent may also be derived, for appropriate prices, but why is this less useful?]
2. Define an equilibrium, stating carefully the objects an equilibrium comprises and the conditions it must satisfy.
3. Show that if the resource constraints are satisfied and the agents’ budget constraints holds with equality in all periods, then the government’s budget balance constraint holds in all periods.
4. Using your result in (b)-(c) to show that, for any given initial tax on capital  $\kappa_0$ , an equilibrium allocation  $(c, L, k)$  must satisfy the resource constraint and,

$$u_c(c_t, L_t)(k_{t+1} + c_t) + u_l(c_t, L_t)L_t = \frac{1}{\beta}u_c(c_{t-1}, L_{t-1})k_t$$

for all  $t = 1, 2, \dots$  and

$$u_c(c_0, L_0)(k_1 + c_0) + u_l(c_0, L_0)(1 - L_0) = \kappa_0(F_k(k_0, L_0) - \delta)k_0.$$

Use this to state the Ramsey problem as a choice over  $(c, L, k)$  for given  $\kappa_0$ .

5. Show that if  $(c_t, L_t, k_t) \rightarrow (c^{ss}, L^{ss}, k^{ss})$  then  $\kappa_t \rightarrow 0$  [Note: make sure you prove any property about Lagrangian multipliers at a steady state]. What happened to a Laffer curve? How can  $g$  be financed by labor income taxes only even if  $g$  is very large?

## 5 Money and the Friedman Rule (Practice problem).

Not for grade.

In this problem we look at the Ramsey problem for a monetary economy. Preferences are given by

$$\sum_{t=0}^{\infty} \beta^t u(c_t, m_t, n_t)$$

where  $c_t$  and  $n_t$  are consumption and labor as before, and  $m_t = M_{t,t+1}/p_t$  is the representative agent's real money balance. Technology is linear:

$$c_t + g_t = n_t$$

The consumer's budget constraint (in terms of goods) is given by:

$$\frac{M_{t,t+1}}{p_t} + \frac{q_{t+1}}{q_t} b_{t+1} + c_t \leq (1 - \tau_t)n_t + b_t + \frac{M_{t-1,t}}{p_t}$$

where  $\frac{q_{t+1}}{q_t}$  is the real price at time  $t$  of a risk-free one-period real bond and  $p_t$  is the real price level.

1. Derive the present value constraint

$$\sum_{t=0}^{\infty} q_t \left[ c_t + \frac{R_{t,t+1}}{1 + R_{t,t+1}} m_t - (1 - \tau_t)n_t \right] \leq \frac{M_{-1,0}}{p_0},$$

where  $R_{t,t+1}$  is the nominal interest rate, defined by

$$1 + R_{t,t+1} = \frac{p_{t+1}}{p_t} \cdot \frac{q_{t+1}}{q_t}.$$

and we normalized so that  $q_0 = 1$ .

2. Define a competitive equilibrium.
3. Prove that the implementability condition

$$\sum_{t=0}^{\infty} \beta^t [u_c(c_t, m_t, n_t)c_t + u_m(c_t, m_t, n_t)m_t + u_n(c_t, m_t, n_t)n_t] = q_0 \frac{M_0}{p_0}$$

and the resource constraints are necessary and sufficient for an allocation to be sustainable by a competitive equilibrium. Set up the Ramsey problem for given  $p_0$ . How will the government choose  $p_0$ ?

4. Suppose preferences are weakly separable

$$u(c, m, n) = U(w(c, m), n)$$

and  $w(c, m)$  is homothetic. Prove that the *Friedman rule* is optimal, that is,  $R_{t,t+1} = 0$  for all  $t = 0, 1, \dots$

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14.161 Advanced Macroeconomics I  
Fall 2012

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