6.003 (Fall 2009)

Quiz #2 *October 28, 2009*

Name:

Kerberos Username:

Please circle your section number:

Section	*Instructor*	*Time*
1	Marc Baldo	10 am
2	Marc Baldo	11 am
3	Elfar Adalsteinsson	1 pm
4	Elfar Adalsteinsson	2 pm

Partial credit will be given for answers that demonstrate some but not all of the important conceptual issues.

Explanations are not required and will not affect your grade.

You have **two hours.**

Please put your initials on all subsequent sheets.

Enter your answers in the boxes.

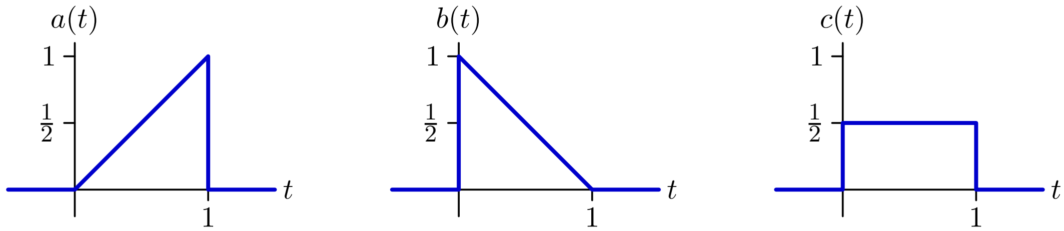
This quiz is closed book, but you may use two 8.5×11 sheets of paper (four sides total).

No calculators, computers, cell phones, music players, or other aids.

1	/18
2	/16
3	/12
4	/20
5	/16
6	/18
Total	/100

1. Convolutions *[18 points]*

Consider the convolution of two of the following signals.

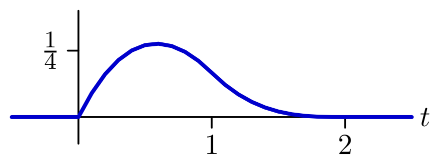


Determine if each of the following signals can be constructed by convolving (a or b or c) with (a or b or c). If it can, indicate which signals should be convolved. If it cannot, put an X in both boxes.

Notice that there are ten possible answers: $(a * a)$, $(a * b)$, $(a * c)$, $(b * a)$, $(b * b)$, $(b * c)$, $(c * a)$, $(c * b)$, $(c * c)$, or (X,X). Notice also that the answer may not be unique.

$\boxed{b}$ $*$ $\boxed{b}$ $=$

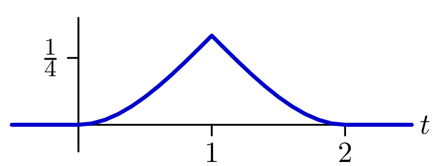
a or b or c or X $\quad$ a or b or c or X



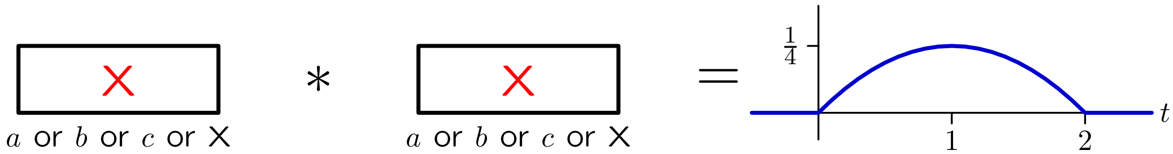
Must be asymmetric with large output at early times and smaller output at later times.

$\boxed{\begin{matrix} a \\ b \end{matrix}}$ $*$ $\boxed{\begin{matrix} b \\ a \end{matrix}}$ $=$

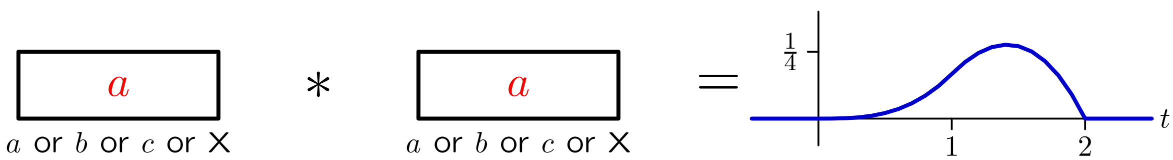
a or b or c or X $\quad$ a or b or c or X



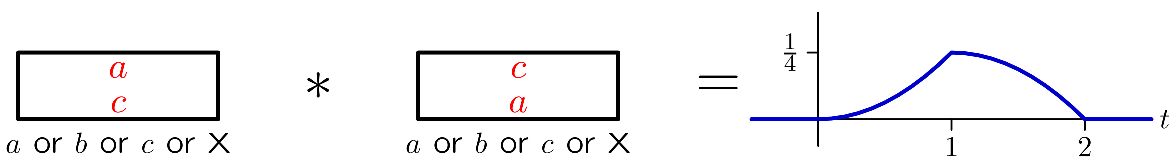
The output is symmetric, which could happen if one of the inputs is a flipped-in-time version of the other input. There are only a few such options. One is $c * c$ — but that would result in a triangle-shaped output. Another symmetric option is $a * b$ (or equivalently $b * a$), which fits with the first interval being concave up.



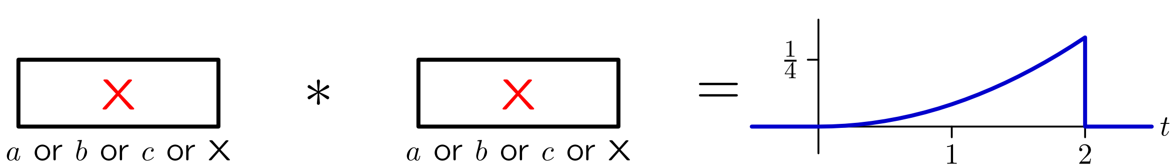
The output is symmetric, which could happen if one of the inputs is a flipped-in-time version of the other input. There are only a few such options. One is $c * c$ — but that would result in a triangle-shaped output. Another symmetric option is $a * b$ (or equivalently $b * a$) — but that would be concave up in first interval. None of the provided functions could result in this output.



Must be asymmetric with small output at early times and larger output at later times.



The first interval looks like the integral of a. The second interval looks like the first interval, but shifted and flipped in time. Thus the answer is $a * c$ or $c * a$.



The step discontinuity in this result could only result if one of the convolved functions contains an impulse.

2. Laplace transforms *[16 points]*

Determine if the Laplace transform of each of the following signals exists. If it does, write **yes** in the box. If it does not, write **no** in the box. If you don't know, write **?** in the box.

Grading: +2 points for each correct answer; −2 points for each incorrect answer; 0 points for each ? or blank response.

$x_1(t) = e^{-t}u(t) + e^{-2t}u(t) + e^{-3t}u(t)$ $X_1(s)$ exists? (**yes** or **no** or **?**): $\boxed{\text{yes}}$

$x_2(t) = e^{-t}u(-t) + e^{-2t}u(t) + e^{-3t}u(t)$ $X_2(s)$ exists? (**yes** or **no** or **?**): $\boxed{\text{yes}}$

$x_3(t) = e^{-t}u(t) + e^{-2t}u(-t) + e^{-3t}u(t)$ $X_3(s)$ exists? (**yes** or **no** or **?**): $\boxed{\text{no}}$

$x_4(t) = e^{-t}u(-t) + e^{-2t}u(-t) + e^{-3t}u(t)$ $X_4(s)$ exists? (**yes** or **no** or **?**): $\boxed{\text{yes}}$

$x_5(t) = e^{-t}u(t) + e^{-2t}u(t) + e^{-3t}u(-t)$ $X_5(s)$ exists? (**yes** or **no** or **?**): $\boxed{\text{no}}$

$x_6(t) = e^{-t}u(-t) + e^{-2t}u(t) + e^{-3t}u(-t)$ $X_6(s)$ exists? (**yes** or **no** or **?**): $\boxed{\text{no}}$

$x_7(t) = e^{-t}u(t) + e^{-2t}u(-t) + e^{-3t}u(-t)$ $X_7(s)$ exists? (**yes** or **no** or **?**): $\boxed{\text{no}}$

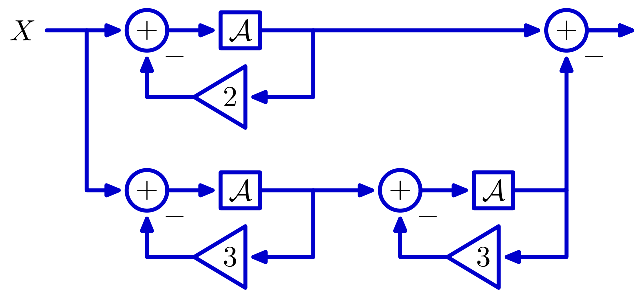
$x_8(t) = e^{-t}u(-t) + e^{-2t}u(-t) + e^{-3t}u(-t)$ $X_8(s)$ exists? (**yes** or **no** or **?**): $\boxed{\text{yes}}$

3. Impulse response *[12 points]*

Sketch a block diagram for a CT system with impulse response

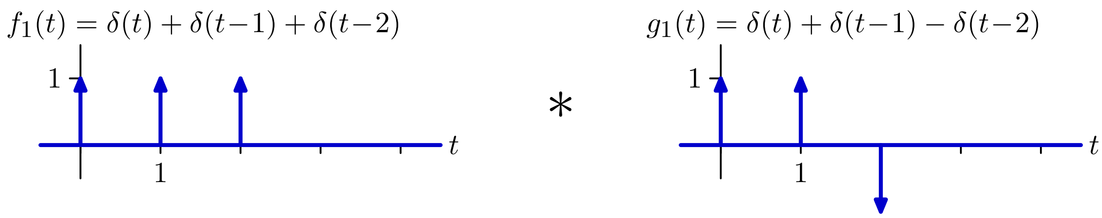
$$h(t) = \left(1 - te^{-t}\right) e^{-2t} u(t)\,.$$

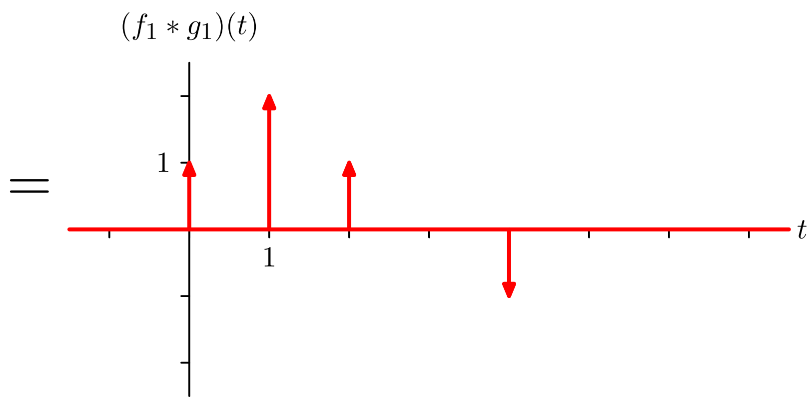
The block diagram should contain only adders, gains, and integrators.



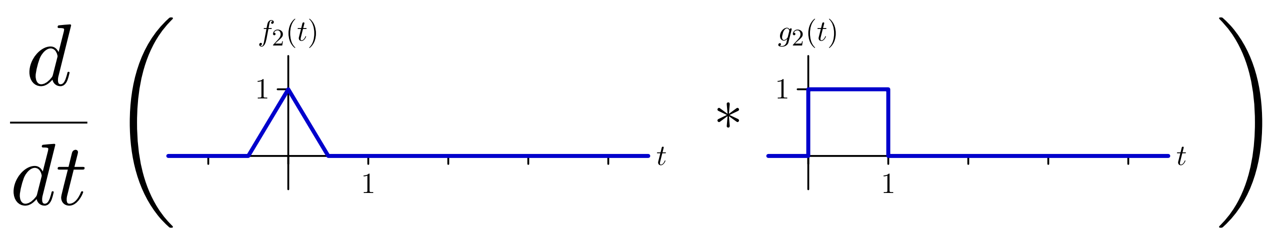
4. Convolutions *[20 points]*

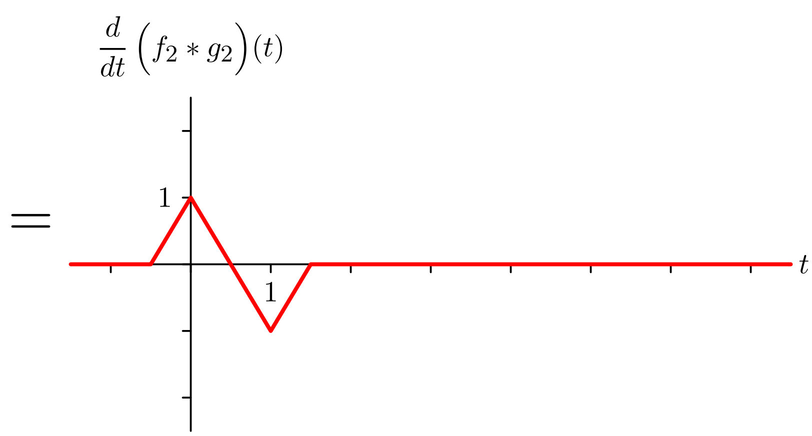
Sketch the signal that results for each of the following parts.



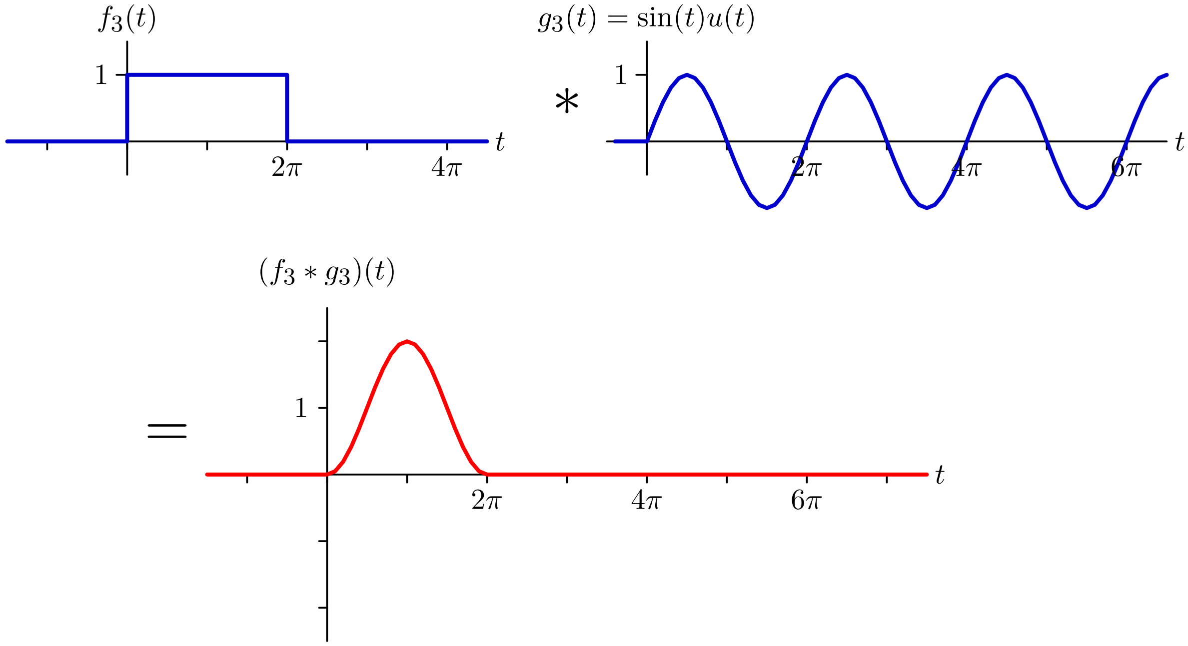


Label the important features of your results!





Label the important features of your results!



Label the important features of your results!

Given

$$f_4[n] = 2^n u[-n] \qquad \text{and} \qquad g_4[n] = \left(\frac{1}{3}\right)^n u[n]$$

enter the following numbers:

$(f_4 * g_4)[-2] =$	$\frac{3}{10}$
$(f_4 * g_4)[-1] =$	$\frac{3}{5}$
$(f_4 * g_4)[0] =$	$\frac{6}{5}$
$(f_4 * g_4)[1] =$	$\frac{2}{5}$
$(f_4 * g_4)[2] =$	$\frac{2}{15}$

5. Z transform *[16 points]*

Let $X(z)$ represent the Z transform of $x[n]$, and let $r_0 < |z| < r_1$ represent its region of convergence (ROC).

Let $x[n]$ be represented as the sum of even and odd parts

$$x[n] = x_e[n] + x_o[n]$$

where $x_e[n] = x_e[-n]$ and $x_o[n] = -x_o[-n]$.

a. Under what conditions does the Z transform of $x_e[n]$ exist?

conditions: $\boxed{\max\left(r_0, \frac{1}{r_1}\right) < \min\left(r_1, \frac{1}{r_0}\right)}$

$$x_e[n] = \frac{x[n] + x[-n]}{2}$$

The Z transform of $y[n] = x[-n]$ is

$$Y(z) = \sum_{n=-\infty}^{\infty} x[-n]z^{-n} = \sum_{l=\infty}^{-\infty} x[l]z^{l} = \sum_{l=-\infty}^{\infty} x[l]\frac{1}{z}^{-l} = X\left(\frac{1}{z}\right)$$

provided $r_0 < |\frac{1}{z}| < r_1$, which is equivalent to $\frac{1}{r_1} < |z| < \frac{1}{r_0}$. The Z transform of $x_e[n]$ will exist if the ROCs of $X(z)$ and $Y(z)$ overlap. Then the ROC for $X_e(z)$ is

$$\max\left(r_0, \frac{1}{r_1}\right) < |z| < \min\left(r_1, \frac{1}{r_0}\right).$$

b. Assuming the conditions given in part a, find an expression for the Z transform of $x_e[n]$, including its region of convergence.

Z transform: $\boxed{\frac{1}{2}\left(X(z)+X\left(\frac{1}{z}\right)\right)}$

ROC: $\boxed{\max\left(r_0,\frac{1}{r_1}\right)<|z|<\min\left(r_1,\frac{1}{r_0}\right)}$

$$X_e(z)=\frac{1}{2}\Big(X(z)+Y(z)\Big)=\frac{1}{2}\left(X(z)+X\left(\frac{1}{z}\right)\right)$$

with the region of convergence given on the previous page.

6. DT approximation of a CT system *[18 points]*

Let H_{C1} represent a **causal** CT system that is described by

$$\dot{y}_C(t) + 3y_C(t) = x_C(t)$$

where $x_C(t)$ represents the input signal and $y_C(t)$ represents the output signal.

$$x_C(t) \rightarrow \boxed{H_{C1}} \rightarrow y_C(t)$$

a. Determine the pole(s) of H_{C1}, and enter them in the box below.

$$\boxed{-3}$$

The the Laplace transform of the differential equation to get

$$sY_C(s) + 3Y_C(s) = X_C(s)$$

and solve for $Y_C(s)/X_C(s) = 1/(s+3)$. The pole is at $s = -3$.

Your task is to design a **causal** DT system H_{D1} to approximate the behavior of H_{C1}.

$$x_D[n] \rightarrow \boxed{H_{D1}} \rightarrow y_D[n]$$

Let $x_D[n] = x_C(nT)$ and $y_D[n] = y_C(nT)$ where T is a constant that represents the time between samples. Then approximate the derivative as

$$\frac{dy_C(t)}{dt} = \frac{y_C(t+T) - y_C(t)}{T}\,.$$

b. Determine an expression for the pole(s) of H_{D1}, and enter the expression in the box below.

$$1 - 3T$$

Take the Z transform of the difference equation

$$\frac{y_D[n+1] - y_D[n]}{T} + 3y_D[n] = x_D[n]$$

to obtain

$$\frac{zY_D(z) - Y_D}{T} + 3Y_D(z) = X_D(z)$$

Solving

$$(z - 1 + 3T)Y_D(z) = TX_D(z)$$

so that

$$H_D(z) = \frac{Y_D(z)}{X_D(z)} = \frac{T}{z - 1 + 3T}\,.$$

There is a pole at $z = 1 - 3T$.

c. Determine the range of values of T for which H_{D1} is stable and enter the range in the box below.

$$0 < T < \tfrac{2}{3}$$

Stability requires that the pole be inside the unit circle

$$-1 < 1 - 3T < 1$$

or

$$-2 < -3T < 0$$

so that

$$0 < T < \frac{2}{3}\,.$$

Now consider a second-order **causal** CT system H_{C2}, which is described by

$$\ddot{y}_C(t) + 100 y_C(t) = x_C(t)\,.$$

d. Determine the pole(s) of H_{C2}, and enter them in the box below.

$$\boxed{\pm j10}$$

Take the Laplace transform of the differential equation to get

$$s^2 Y_C + 100 Y_C = X_C$$

and solve for $Y_C/X_C = 1/(s^2+100)$. There are poles at $s = \pm j10$.

Design a **causal** DT system H_{D2} to approximate the behavior of H_{C2}. Approximate derivatives as before:

$$\dot{y}_C(t) = \frac{dy_C(t)}{dt} = \frac{y_C(t+T) - y_C(t)}{T} \quad \text{and}$$

$$\frac{d^2 y_C(t)}{dt^2} = \frac{\dot{y}_C(t+T) - \dot{y}_C(t)}{T}\,.$$

e. Determine an expression for the pole(s) of H_{D2}, and enter the expression in the box below.

$1 \pm j10T$

$$\frac{d^2 y_C(t)}{dt^2} = \frac{\dot{y_C}(t+T) - \dot{y_C}(t)}{T} = \frac{\frac{y_C(t+2T)-y_C(t+T)}{T} - \frac{y_C(t+T)-y_C(t)}{T}}{T}$$
$$= \frac{y_C(t+2T) - 2y_C(t+T) + y_C(t)}{T^2}\,.$$

Substituting to find the difference equation, we get

$$\frac{y_D[n+2] - 2y_D[n+1] + y_D[n]}{T^2} + 100 y_D[n] = x_D[n]\,.$$

Take the Z transform to find that

$$(z^2 - 2z + 1 + 100T^2) Y_D(z) = T^2 X_D(z)$$

or

$$\frac{Y_D(z)}{X_D(z)} = \frac{T^2}{z^2 - 2z + 1 + 100T^2}\,.$$

The poles are at

$$z = 1 \pm \sqrt{1 - 1 - 100T^2} = 1 \pm j10T$$

f. Determine the range of values of T for which H_{D2} stable and enter the range in the box below.

none

The poles are always outside the unit circle. The system is always unstable.