MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.011 Introduction to Communication, Control and Signal Processing
Fall 2003

FIRST EVENING EXAM

Wednesday October 15, 7:30 PM – 9:30 PM

- Put your name on **each** page of this booklet. Specify your recitation instructor and your recitation time.
- This is a closed book exam, but two $8\frac{1}{2}'' \times 11''$ sheets of notes (both sides) are allowed. They can be as big as $8\frac{1}{2}'' \times 11''$ or as small as you'd like and you can write on one side or two sides of each, but only two sheets are allowed.
- Everything on the notes must be in your original handwriting (i.e., material cannot be xeroxed from solutions, tables, books, etc).
- You have two hours for this exam.
- Calculators are NOT allowed.
- We will NOT provide a table of transforms.
- There are 4 problems on the exam with the percentage for each part and the total percentage for each problem as indicated. Note that the problems do not all have the same total percentage.
- Make sure you have seen all 14 numbered sides of this answer booklet.
- The problems are not in order of difficulty. We recommend that you read through all the problems, then do the problems in whatever order suits you best.
- We tried to provide ample space for you to write in. However, the space provided is not an indication of the length of the explanation required. Short, to the point, explanations are preferred to long ones that show no understanding.
- Please be neat—we cannot grade what we cannot decipher.

All work and answers must be in the space provided on the exam booklet. You are welcome to use scratch pages that we provide but when you hand in the exam we will not accept any pages other than the exam booklet. There will be absolutely no exceptions.

Exam Grading

In grading all of the 6.011 exams we will be focusing on your level of understanding of the material associated with each problem. When we grade each part of a problem we will do our best to assess, from your work, your level of understanding. On each part of an exam question we will also indicate the percentage of the total exam grade represented by that part, and your numerical score on the exam will then be calculated accordingly.

Our assessment of your level of understanding will be based upon what is given in your solution. A correct answer with no explanation will not receive full credit, and may not receive much–if any. An incorrect final answer having a solution and explanation that shows excellent understanding quite likely will receive full (or close to full) credit.

This page is intentionally left blank. Use it as scratch paper.
No work on this page will be evaluated.

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6.011 Introduction to Communication, Control and Signal Processing
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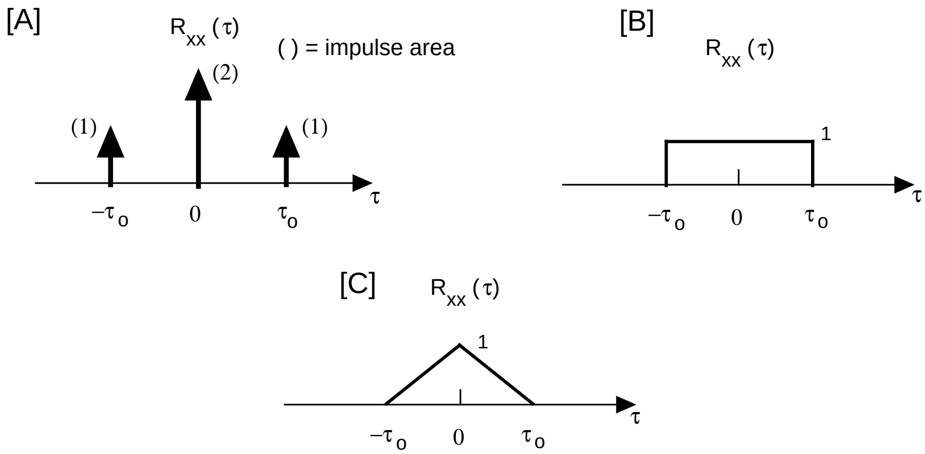
Full Name:____

	Points	Grader
1		
2(a)		
2(b)		
2(c)		
2(d)		
3(a)		
3(b)		
3(c)		
4(a)		
4(b)		
4(c)		
4(d)		
Total		

Problem 1 (10%)

The figure below shows three candidates (labeled [A], [B], and [C]) for the autocorrelation function, $R_{xx}(\tau)$, of a wide-sense stationary, continuous-time random process $x(t)$.



For each candidate $R_{xx}(\tau)$, state whether or not it is a possible autocorrelation function for a wide-sense stationary random process $x(t)$. Give **brief** justifications for your answers.

Candidate [**A**]:	**POSSIBLE**	**NOT POSSIBLE**
Candidate [**B**]:	**POSSIBLE**	**NOT POSSIBLE**
Candidate [**C**]:	**POSSIBLE**	**NOT POSSIBLE**

Brief justification:

Problem 2 (30%)

Consider a continuous-time random process $x(t)$ defined as follows:

$$x(t) = \cos(\omega t + \theta), \quad \text{for } -\infty < t < \infty,$$

where ω and θ are statistically independent random variables, with ω being uniformly distributed on the interval $[-\omega_o, \omega_o]$ and θ being uniformly distributed on the interval $[0, 2\pi]$.

The triginometric identity below might be helpful:

$$\cos(A)\cos(B) = \frac{\cos(A+B) + \cos(A-B)}{2}.$$

(a) (10%) Determine the following ensemble-average statistics of the random process $x(t)$:

(i) The mean function $\mu_x(t) \equiv E[x(t)]$.

$\mu_x(t) =$

Work to be looked at:

(ii) The correlation function $R_{xx}(t_1, t_2) \equiv E[x(t_1)x(t_2)]$.

$R_{xx}(t_1, t_2) =$

Work to be looked at:

(b) (10%) Determine in terms of ω and θ the following time averages of a single realization (outcome) of the random process $x(t)$.

$$\langle x(t) \rangle \equiv \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, dt$$

$$\langle x(t+\tau_o)x(t) \rangle \equiv \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t+\tau_o)x(t)\, dt, \quad \text{where } \tau_o \text{ is a positive constant}$$

(i) $\langle x(t) \rangle =$

Work to be looked at:

(i) cont.

More work to be looked at:

(ii) $\langle x(t+\tau_o)x(t)\rangle =$

Work to be looked at:

(c) (5%) Is the process wide-sense stationary? Circle your answer and clearly state your reasoning.

YES **NO**

Reasoning:

(d) (5%) Can a really long record of a single realization be used with appropriate time averaging to calculate (at least approximately) $R_{xx}(t_1, t_2)$? Circle your answer and clearly state your reasoning.

YES **NO**

Reasoning:

Problem 3 (30%)

An A-to-D converter can basically be represented as a C/D converter followed by a quantizer. As considered in Problem 4.4 (LN 8.7), a useful model for quantization error in a linear quantizer is to represent the error as a zero-mean i.i.d. process with variance σ_e^2.
Consequently a useful model for an A/D converter is shown in Figure 3-1.

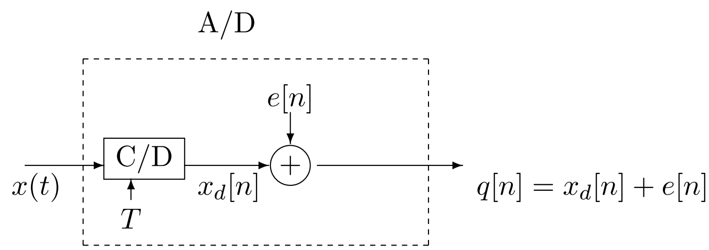


Figure 3-1:

$q[n]$ represents the quantized signal and $e[n]$ represents the error introduced by quantization. Assume

1) $x(t)$ is a zero mean WSS random process
2) $e[n]$ is i.i.d, zero-mean, with variance $\sigma_e^2 = \frac{1}{5} \times 10^{-3}$
3) $x_d[n]$ and $e[n]$ are statistically independent random processes.
4) The PSD, $S_{xx}(j\omega)$, of $x(t)$ is as shown in Figure 3-2.
5) The sampling period is $T = \frac{1}{4} \times 10^{-5}$, i.e. the sampling frequency $\omega_s = \frac{2\pi}{T} = 8\pi \times 10^5$.

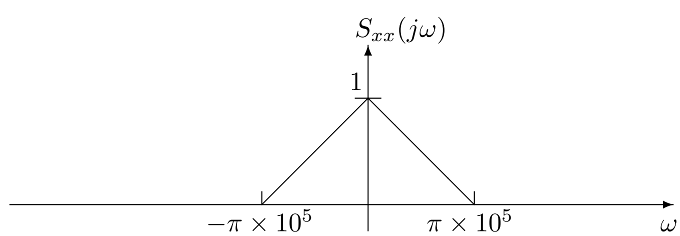
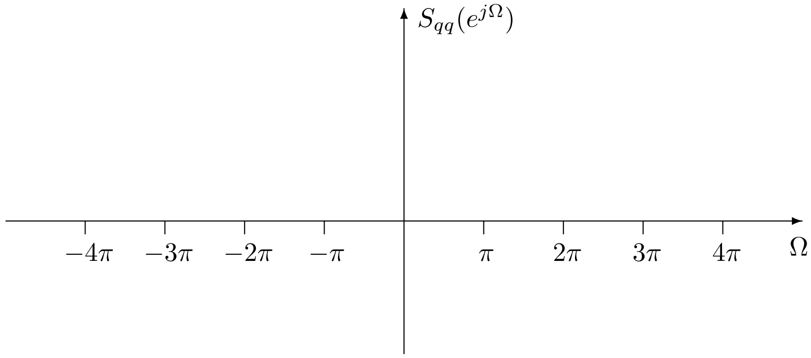


Figure 3-2: Power spectral density of $x(t)$.

(a) (8%) Determine and make a labeled sketch of $S_{qq}(e^{j\Omega})$, the PSD of $q[n]$, in the range $|\Omega| \leq 4\pi$.



$S_{qq}(e^{j\Omega}) =$

Work to be looked at:

(b) (8%) The signal-to-noise ratio, SNR_q, of the quantized signal $q[n]$ is defined as:

$$\mathrm{SNR}_q \triangleq \frac{E\{x_d^2[n]\}}{E\{e^2[n]\}}$$

Determine SNR_q.

$\mathrm{SNR}_q =$

Work to be looked at:

(c) (14%) $q[n]$ is now processed as shown in Figure 3-3.

$$q[n] \longrightarrow \boxed{H(e^{j\Omega})} \longrightarrow r[n] = x_r[n] + e_r[n]$$

Figure 3-3:

where $H(e^{j\Omega})$ is the ideal lowpass filter shown in Figure 3-4, $x_r[n]$ is due only to $x_d[n]$, and $e_r[n]$ is due only to $e[n]$.

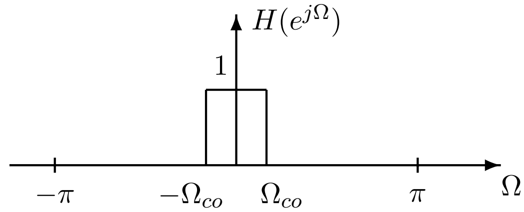


Figure 3-4:

What value of Ω_{co} would you choose so that $R_{x_r x_r}[n] = R_{x_d x_d}[n]$ and $E\{x_d^2[n]\}/E\{e_r^2[n]\}$ (i.e. the SNR after filtering) is maximized. State the maximized SNR value.

$\Omega_{co} =$ maximized SNR=

Work to be looked at:

Problem 4 (30%)

Consider the following causal discrete-time state-space system

$$
\begin{aligned}
\mathbf{q}[n+1] &= \mathbf{A}\mathbf{q}[n] + \mathbf{b}x[n] \\
y[n] &= \mathbf{c}^T\mathbf{q}[n] + dx[n]
\end{aligned}
$$

where

$$
\mathbf{A} = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ 0 & 2 \end{bmatrix}
$$

and $\mathbf{b}$, $\mathbf{c}^T$, and d are unknown.

(a) (5%) Determine the eigenvalues λ_1, λ_2 and the eigenvectors $\mathbf{v_1}$, $\mathbf{v_2}$ of $\mathbf{A}$.

$\lambda_1 =$ $\qquad\qquad$ $\mathbf{v_1} =$

$\lambda_2 =$ $\qquad\qquad$ $\mathbf{v_2} =$

Work to be looked at:

(b) (7%) Specify all the possible initial conditions (i.e. $\mathbf{q}[0]$) such that the ZIR will decay asymptotically to zero.

$\mathbf{q}[0] =$

(c) (8%) Indicate which, if any, of the following could be ZSR system functions for the system.

(i) $H(z) = \frac{z^{-1}}{1-\frac{1}{3}z^{-1}}$

Is it a possible ZSR system function? Circle your answer:

YES **NO**

Explanation:

(ii) $H(z) = \frac{1+\frac{1}{2}z^{-1}}{1-\frac{1}{2}z^{-1}}$

Is it a possible ZSR system function? Circle your answer:

YES **NO**

Explanation:

(d) (10%) We want to describe the system in terms of the new set of state variables $\mathbf{f}[n] = \begin{bmatrix} f_1[n] \\ f_2[n] \end{bmatrix}$, where $f_1[n] = q_1[n] + q_2[n]$ and $f_2[n] = q_1[n] - q_2[n]$.
In other words, we want to describe the system using the following equations:

$$\begin{aligned} \mathbf{f}[n+1] &= \tilde{\mathbf{A}}\mathbf{f}[n] + \tilde{\mathbf{b}}x[n] \\ y[n] &= \tilde{\mathbf{c}}^T\mathbf{f}[n] + \tilde{d}x[n] \end{aligned}$$

(i) Determine $\tilde{\mathbf{A}}$, $\tilde{\mathbf{b}}$, $\tilde{\mathbf{c}}^T$, and $\tilde{d}$ in terms of $\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}^T$, and d.

$\tilde{\mathbf{A}} =$ $\tilde{\mathbf{b}} =$ $\tilde{\mathbf{c}}^T =$ $\tilde{d} =$

Work to be looked at:

(ii) We can express the ZIR of the system in the general form $\mathbf{f}[n] = \alpha_1 \tilde{\lambda}_1^n \tilde{\mathbf{v}}_\mathbf{1} + \alpha_2 \tilde{\lambda}_2^n \tilde{\mathbf{v}}_\mathbf{2}$, where α_1 and α_2 are constants.
Determine $\tilde{\lambda}_1$, $\tilde{\lambda}_2$, $\tilde{\mathbf{v}}_\mathbf{1}$, and $\tilde{\mathbf{v}}_\mathbf{2}$.

$\tilde{\lambda}_1 =$ $\tilde{\lambda}_2 =$ $\tilde{\mathbf{v}}_\mathbf{1} =$ $\tilde{\mathbf{v}}_\mathbf{2} =$

Work to be looked at: