Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.011: Introduction to Communication, Control and Signal Processing

QUIZ 2, Spring 2003

Solutions

- This quiz is **closed book**, but **two** "crib" sheets are allowed.
- Put your name on **each** sheet of the answer booklet, and your recitation instructor's name and time on the cover page of that booklet.
- **Only** the answer booklet will be considered in the grading; **no additional answer or solution written elsewhere will be considered.** Absolutely no exceptions!
- **Neat work and clear explanations count; show all relevant work and reasoning!** You may want to first work things through on scratch paper and then neatly transfer to the answer booklet the work you would like us to look at. Let us know if you need additional scratch paper.
- The quiz will be graded out of **50 points**. The **three problems** are nominally **weighted as indicated** (but our legal department wishes to let you know that we reserve the right to modify the weighting *slightly* when we grade, if your collective answers and common errors end up suggesting that a modified weighting would be more appropriate).

Problem 1 (36 points)

(a) (i) Using the usual notation for state space model, we have $H(s) = \mathbf{c^T}(\mathbf{sI} - \mathbf{A})^{-1}\mathbf{b} + \mathbf{d}$. For the forst order case, this reduces to $H(s) = \frac{cb}{s-a} + d$. Comparing with $H_1(s) = \frac{1}{s-1} + 1$. We find that

$$\begin{aligned} \dot{q}_1(t) &= q_1(t) + x_1(t) \\ y(t) &= q_1(t) + x_1(t) \end{aligned}$$

(ii) Is your state-space model for System 1:

- reachable? Yes. The first order system is in modal form and $\beta = 1$. Alternatively there is no hidden mode.
- observable? Yes. Using the same reason as above, since $\xi = 1 \neq 0$The system is observable. Alternatively there is no hidden mode..
- asymptotically stable? No. The system has a pole at 1, in the right half plane.

(b) (i) Using $H_2(s) = \frac{cb}{s-a} + d$ and comparing with the state space equations we get,$H_2(s) = \frac{2}{s-\mu}$

(ii) For what values of μ, if any, is the state-space model of System 2:

- unreachable? No values. Note that $\beta = 1$ for any choice of μ. Alternatively the system has no hidden modes.
- unobservable? No values. Note that $\xi = 1$ for any choice of μ. Alternatively the system has no hidden mode.
- asymptotically stable? $\mu < 0$. Note that we are assuming that μ is real since it is a parameter of the state space model.

(c) (i)

$$\begin{aligned} \dot{q}_1(t) &= q_1(t) + x_1(t) \\ \dot{q}_2(t) &= \mu q_2(t) + x_2(t) \\ &= \mu q_2(t) + q_1(t) + x_1(t) \end{aligned}$$

$$y_2(t) = 2q_2(t)$$

Combining, these we get:

$$\dot{\mathbf{q}}(t) = \mathbf{A}\mathbf{q}(t) + \mathbf{b}x_1(t)$$

Combine the state-space models in (a) and (b) to obtain a second-order state-space model of the form

$$\dot{\mathbf{q}}(t) = \begin{bmatrix} 1 & 0 \\ 1 & \mu \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & \mu \end{bmatrix} \mathbf{q}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_1(t)$$

$$y(t) = \begin{bmatrix} 0 & 2 \end{bmatrix} q(t)$$

(ii) Since we have solved from the state space equations in the previous part, it is easy to compute the transfer function. Using the usual notation:

$$\begin{aligned}
H(s) &= \mathbf{c}^T(s\mathbf{I}-\mathbf{A})^{-\mathbf{1}}\mathbf{b}+\mathbf{d} \\
&= \begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} s-1 & 0 \\ -1 & s-\mu \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \frac{2s}{(s-1)(s-\mu)} \begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} s-\mu & 0 \\ 1 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \frac{2s}{(s-1)(s-\mu)}
\end{aligned}$$

Note that $H(s) = H_1(s)H_2(s)$. This is expected because the overall system is a cascade of two LTI systems.

(iii) Since $\mathbf{A}$ is triangular, its eigen values are same as the elements on its principal diagonal. accordingly, we have

$$\lambda_1 = 1, \lambda_2 = \mu$$

Note this makes sense since these are also the poles of $H(s)$. Let $\mathbf{v_1}$ and $\mathbf{v_2}$ be the associated eigen vectors. Using

$$(\lambda_1\mathbf{I}-\mathbf{A})\mathbf{v_1} = \mathbf{0}, \mathbf{v_1} = \mathbf{k_1}\begin{bmatrix} 1-\mu \\ 1 \end{bmatrix}$$

$$(\lambda_2\mathbf{I}-\mathbf{A})\mathbf{v_2} = \mathbf{0}, \mathbf{v_2} = \mathbf{k_2}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Here k_1 and k_2 can be arbitrary non-zero constants.
We can write the state evolution equation as

$$\mathbf{q(t)} = \alpha_\mathbf{1}\mathbf{v_1(t)e^t} + \alpha_\mathbf{2}\mathbf{v_2e^{\mu}t}$$

Where α_1 and α_2 are determined by initial conditions. Since we want the response to decay to zero, we require that the first mode should not be excited and that $\mu < 0$. The initial condition thus is of the form $\mathbf{q(0)} = \alpha \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Where $\alpha \neq 0$.

(iv) For what values of μ, if any, is the overall system in (c)(ii):

- unreachable? — which natural frequencies are unreachable?
  The system is unreachable if $\beta_i = 0$ for some β_i. Since our system is second order, we can write

$$\mathbf{b} = \beta_\mathbf{1}\begin{bmatrix} 1-\mu \\ 1 \end{bmatrix} + \beta_\mathbf{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

  If $\mu = 0$, the clearly $\beta_2 = 0$ and the system is unreachable. The second mode will be unreachable.

- unobservable? — which natural frequencies are unobservable?
  The system in unobservable if $\xi_i = 0$ for some ξ_i.

$$\xi_1 = \mathbf{c}^T \mathbf{v_1} = \left[\begin{array}{cc} 0 & 2 \end{array}\right] \left[\begin{array}{c} 1-\mu \\ 1 \end{array}\right] = 2 \neq 0$$

and

$$\xi_2 = \mathbf{c}^T \mathbf{v_2} = \left[\begin{array}{cc} 0 & 2 \end{array}\right] \left[\begin{array}{c} 0 \\ 1 \end{array}\right] = 2 \neq 0$$

Clearly the system is not unobservable for any value of μ

For $\mu = 0$, $H_2(s) = \frac{2}{s}$, but $H(s) = \frac{s}{s-1}\frac{2}{s} = \frac{2}{s-1}$. Thus there is a hidden mode for $\mu = 0$. The pole of the second subsystem is shielded from the input by the zero of the first subsystem. Hence the mode at $\lambda_2 = 0$ is unreachable. But it is still visible in the output. So the system is observable.

(d) Since we need to find the natural frequencies, we need to set up the characteristic equation of the closed loop response and solve for the eigen values.

$$\mathbf{A} + \mathbf{b}\mathbf{g}^{\mathbf{T}} = \left[\begin{array}{cc} 1 & 0 \\ 1 & \mu \end{array}\right] + \left[\begin{array}{c} 1 \\ 1 \end{array}\right] \left[\begin{array}{cc} g_1 & g_2 \end{array}\right] = \left[\begin{array}{cc} 1+g_1 & g_2 \\ 1+g_1 & \mu+g_2 \end{array}\right]$$

The characteristic polynomial is obtained from $det(s\mathbf{I} - \mathbf{A} + \mathbf{b}\mathbf{g}^T)$. After some simplification this reduces to $s^2 - s(g_1 + g_2 + \mu + 1) + \mu(1 + g1)$ Comparing with our desired characteristic funcion $(s+1)^2 + 1 = s^2 + 2s + 2\mathrm{m}$, we get

$$\begin{aligned} g_1 + g_2 &= -2 - \mu - 1 \\ \mu(1 + g_1) &= 2 \end{aligned}$$

Solving these simulataneous equations, we have

$$\begin{aligned} g_1 &= \frac{2}{\mu} - 1 \\ g_2 &= -2 - \mu - \frac{2}{\mu} \end{aligned}$$

Again, if $\mu = 0$, the value of these gains become infinite. This reconciles with the fact that the system in unreachable for $\mu = 0$, and feedback can no longer stabilize the system.

(e) We need to find an observer such that $\mathbf{A} + l\mathbf{c}^T$, has eigen values at $-\frac{1}{0.5}$ and $-\frac{1}{0.25}$.

$$\begin{aligned} det(s\mathbf{I} - \mathbf{A} + l\mathbf{c}^T) &= \left[\begin{array}{cc} s-1 & -2l_1 \\ -1 & s-(\mu+2l_2) \end{array}\right] \\ &= s^2 - s(1 + \mu + 2l_2) + (\mu + 2l_2 - 2l_1) \end{aligned}$$

Comparing the coefficients with $(s+2)(s+4) = s^2+6s+8$, we get $l_1 = -\frac{15}{2}$ and $l_2 = -\frac{7+\mu}{2}$. This scheme will work for all values of μ. This is consistent with the fact that the system is observable for all values of μ.

Problem 2 (7 points)

Consider a nonlinear time-invariant state-space model described in the form

(a) For the equilibrium points we will set $\dot{q_1}(t) = 0$ and $\dot{q_2}(t) = 0$. Solving, we get $q_2(t) = 0$ and $\beta q_1^3(t) = \overline{x}$. Thus the equilibrium points are given by $\overline{q_1} = (\frac{\overline{x}}{\beta})^{\frac{1}{3}}$ and $\overline{q_2} = 0$.

(b) We linearize the first state space equation as follows

$$\begin{aligned}\dot{q_1})(t) &= q_2(t)\\ \dot{\overline{q_1}} + \dot{\check{q}}_1(t) &= \overline{q_2} + \check{q}_2(t)\\ \dot{\check{q}}_1(t) &= \check{q}_2(t)\end{aligned}$$

The second state equation may be linearized as follows,

$$\begin{aligned}\dot{q_2}(t) &= -\beta q_1^3(t) + x(t)\\ \dot{\overline{q_2}} + \dot{\check{q}}_2(t) &= -\beta(\overline{q_1} + \check{q}_1(t))^3 + \overline{x} + \check{x}(t)\\ \dot{\check{q}}_2(t) &= \overline{q_1}^3(1 + \frac{3\check{q}_1(t)}{\overline{q_1}}) + \overline{x} + \check{x}(t)\\ &= -3\beta\overline{q_1}^2\check{q}_1(t) + \check{x}(t)\end{aligned}$$

Combining, we get $\dot{\check{\mathbf{q}}}(t) = \begin{bmatrix} 0 & 1 \\ -3\beta\overline{q}^2 & 0 \end{bmatrix}\check{\mathbf{q}}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\check{x}(t)$

The natural frequencies of the system are at $j(3\beta\overline{q}_1^2)^{\frac{1}{2}}$ and $-j(3\beta\overline{q}_1^2)^{\frac{1}{2}}$. Since these are not in the left half plane, the system is not asymptotically stable.

Problem 3 (7 points)

It was announced in class that $0 < \Omega_0 < \pi$.

The signals that produce zero $y_c(t)$ are those for which ω_{in} maps to $\underline{+}\Omega_o + 2k\pi$ for integer k. Thus we have

$$\omega_{in} = \underline{+}\frac{\Omega_o}{T} + \frac{2k\pi}{T}$$

Also note that

$$\begin{aligned}cos(\omega_{in}t + \theta)|_{t=nT} &= cos(\omega_{in}Tn + \theta)\\ &= cos(\underline{+}\Omega_o n + \theta + 2k\pi)\\ &= cos(\underline{+}\Omega_o n + \theta) = x[n]\end{aligned}$$

With this $x[n]$, the notch filter produces $y[n] = 0$, so $y_c(t) = 0$. No other ω_{in} will have this property of aliasing to the appropriate locations.