Quiz 2 - Basic Equations of Electrodynamics

Mathematical Identities

$v(t) = R_e\{\underline{V}e^{j\omega t}\}$ where $\underline{V} = |\underline{V}|e^{j\phi}$

$\nabla = \hat{x}\,\partial/\partial x + \hat{y}\,\partial/\partial y + \hat{z}\,\partial/\partial z$

$\overline{A}\cdot\overline{B} = A_xB_x + A_yB_y + A_zB_z$

$\nabla^2\phi = (\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2)\phi$

$\sin^2\theta + \cos^2\theta = 1$

$\nabla\cdot(\nabla\times\overline{A}) = 0$

$\nabla\times(\nabla\times\overline{A}) = \nabla(\nabla\cdot\overline{A}) - \nabla^2\overline{A}$

$\int_V(\nabla\cdot\overline{G})dv = \oint_S\overline{G}\cdot\hat{n}da$

$\int_S(\nabla\times\overline{G})\cdot\hat{n}da = \oint_C\overline{G}\cdot d\hat{s}$

$e^{j\omega t} = \cos\omega t + j\sin\omega t$

$\cos\alpha + \cos\beta = 2\cos[(\alpha+\beta)/2]\cos[(\alpha-\beta)/2]$

$\underline{H}(\omega) = \int_{-\infty}^{+\infty}h(t)e^{-j\omega t}dt$

$e^x = 1 + x + x^2/2! + x^3/3! + \ldots$

$\sin\alpha = (e^{j\alpha} - e^{-j\alpha})/2j$

$\cos\alpha = (e^{j\alpha} + e^{-j\alpha})/2$

Electromagnetic Variables

$\overline{E}$ = electric field (V/m)

$\overline{H}$ = magnetic field (A/m)

$\overline{D}$ = electric displacement (C/m^2)

$\overline{B}$ = magnetic flux density (T)

Tesla (T) = Weber/m^2 = 10^4 gauss

ρ = charge density (C/m^3)

$\overline{J}$ = current density (A/m^2)

σ = conductivity (Siemens/m)

$\overline{J}_s$ = surface current density (A/m)

ρ_s = surface charge density (C/m^2)

Boundary Conditions

$\hat{n}\times(\overline{E}_1 - \overline{E}_2) = 0$

$\hat{n}\times(\overline{H}_1 - \overline{H}_2) = \overline{J}_s$

$\hat{n}\cdot(\overline{B}_1 - \overline{B}_2) = 0$

$\hat{n}\cdot(\overline{D}_1 - \overline{D}_2) = \rho_s$

$\underline{\overline{E}} = \underline{\overline{H}} = 0$ *if* $\sigma = \infty$

Maxwell's Equations, Force

$\nabla\times\overline{E} = -\partial\overline{B}/\partial t$

$\oint_C\overline{E}\cdot d\hat{s} = -\frac{d}{dt}\int_S\overline{B}\cdot\hat{n}da$

$\nabla\times\overline{H} = \overline{J} + \partial\overline{D}/\partial t$

$\oint_C\overline{H}\cdot d\bar{s} = \int_S\overline{J}\cdot\hat{n}da + \frac{d}{dt}\int_S\overline{D}\cdot\hat{n}da$

$\nabla\cdot\overline{D} = \rho \rightarrow \oint_s\overline{D}\cdot\hat{n}da = \int_V\rho dv$

$\nabla\cdot\overline{B} = 0 \rightarrow \oint_S\overline{B}\cdot\hat{n}da = 0$

$\nabla\cdot\overline{J} = -\partial\rho/\partial t$

$\overline{f} = q(\overline{E} + \overline{v}\times\mu_o\overline{H})$ [N]

Waves

$(\nabla^2 - \mu\varepsilon\,\partial^2/\partial t^2)\overline{E} = 0$

$(\nabla^2 + k^2)\,\underline{\overline{E}} = 0,\quad \underline{\overline{E}} = \underline{\overline{E}}_o e^{-j\overline{k}\cdot\overline{r}}$

$k = \omega\sqrt{\mu\varepsilon} = \frac{\omega}{c} = \frac{2\pi}{\lambda} = \sqrt{k_x^2 + k_y^2 + k_z^2}$

$v_p = \omega/k,\quad v_g = \partial\omega/\partial k$

$E_x(z,t) = E_+(z-ct) + E_-(z+ct)$ [or $(\omega t-kz)$ or $(t-z/c)$]

$H_y(z,t) = (1/\eta_o)[E_+(z-ct) - E_-(z+ct)]$

$E_x(z,t) = R_e\{\underline{E}_x(z)e^{j\omega t}\}$

$<\overline{E}\times\overline{H}> = \frac{1}{2}Re\{\underline{\overline{E}}\times\underline{\overline{H}}^*\}$

$$\oint_s(\overline{E}\times\overline{H})\cdot\hat{n}da = -\frac{d}{dt}\int_v\left(\frac{1}{2}\varepsilon\left|\overline{E}\right|^2 + \frac{1}{2}\mu\left|\overline{H}\right|^2\right)dv - \int_v\overline{E}\cdot\overline{J}dv$$

Constants

$\varepsilon_o = 8.85\times10^{-12}$ F/m

$\mu_o = 4\pi\times10^{-7}$ H/m

$c = 1/\sqrt{\varepsilon_o\mu_o} \cong 3\times10^8$ m/s

$h = 6.624\times10^{-34}$ Js

$e = 1.60\times10^{-19}$ [C]

$k_B = 1.38\times10^{-23}$ J/K

$\eta_o \cong 377\text{ohms} = \sqrt{\mu_o/\varepsilon_o}$

$m_e = 9.1066\times10^{-31}$ kg

Media

$\overline{D} = \varepsilon\overline{E} = \varepsilon_o\overline{E} + \overline{P}$

$\nabla\cdot\overline{D} = \rho_f$

$\nabla\cdot\varepsilon_o\overline{E} = \rho_f + \rho_p$

$\underline{\overline{D}} = \underline{\overline{\overline{\varepsilon}}}\,\underline{\overline{E}},\quad \overline{J} = \sigma\overline{E}$

$\overline{B} = \mu\overline{H} = \mu_o(\overline{H} + \overline{M})$

$\varepsilon = \varepsilon_o(1 - \omega_p^2/\omega^2)$

$\omega_p = \sqrt{Ne^2/m\varepsilon_o}$

$\underline{\varepsilon}_{eff} = \varepsilon(1 - j\sigma/\omega\varepsilon)$

$\Delta = 2/(\sigma\eta)$

$\delta = \sqrt{2/\omega\mu\sigma}$

Planar Interfaces

$\theta_i = \theta_r$

$\frac{\sin\theta_t}{\sin\theta_i} = \frac{k_i}{k_t} = \frac{\sqrt{\varepsilon_i\mu_i}}{\sqrt{\varepsilon_t\mu_t}} \triangleq \frac{n_i}{n_t}$

$\theta_c = \sin^{-1}(n_t / n_i)$

$\theta_B = \tan^{-1}\left(n_t / n_i\right)$

$1 + \underline{\Gamma} = \underline{T}$

$\underline{\Gamma}_{TE/TM} = (Z_n^{TE/TM} - 1)/(Z_n^{TE/TM} + 1)$

$Z_n^{TE} = \frac{\eta_t \cos\theta_i}{\eta_i \cos\theta_t}$

$Z_n^{TM} = \frac{\eta_t \cos\theta_t}{\eta_i \cos\theta_i}$

$k = k' + jk''$

$P_d \cong |\underline{J}_s|^2 / 2\sigma\delta \ [W/m^2]$

Waveguides

$\underline{\overline{E}}_{TE} = \hat{y}E_o \sin k_x x \cdot e^{-jk_z z}$

$\underline{\overline{E}}_{TE} = \hat{y}E_o \sin k_x x \cdot e^{-\alpha z}$

$k_x^2 + k_z^2 = k_o^2 = \omega^2\mu\varepsilon$

$1/\lambda_g = 1/\lambda_z = \sqrt{1/\lambda_o^2 - 1/\lambda_x^2}$

$v_p = \frac{\omega}{k}, \ v_g = \frac{d\omega}{dk}$

Quasistatics

$\overline{E} = -\nabla\Phi$

$\nabla^2\Phi = -\rho/\varepsilon_o$

$\underline{\Phi}(\overline{r}) = \int_{V'} \{\underline{\rho}(\overline{r}')/4\pi\varepsilon|\overline{r}' - \overline{r}|\} dv'$

$\mu_o\overline{H} = \nabla \times \overline{A}$

$\nabla^2\overline{A} = -\mu_o\overline{J}$

$\underline{\overline{A}}(r) = \int_{V'} \{\mu_o \underline{\overline{J}}(\overline{r}')/4\pi|\overline{r}' - \overline{r}|\} dv'$

TEM Transients

$\frac{dv(z,t)}{dz} = -L\frac{di(z,t)}{dt}$

$\frac{d^2v(z,t)}{dz^2} = LC\frac{d^2v(z,t)}{dt^2}$

$v(z,t) = f_+(t - \frac{z}{c}) + f_-(t + \frac{z}{c})$

$i(z,t) = Y_o\left(f_+(t - \frac{z}{c}) - f_-(t + \frac{z}{c})\right)$

$c = 1/\sqrt{LC} = 1/\sqrt{\mu\varepsilon}$

$Z_o = \sqrt{L/C}$

$\Gamma_L = \frac{R_L - Z_o}{R_L + Z_o} = \frac{R_{Ln} - 1}{R_{Ln} + 1}$

$v_{Th} = 2f_+(t), \ R_{Th} = Z_o$

Circuit Elements

$C = \frac{Q}{V}$

$L = \frac{\Lambda}{I}$

$i(t) = C\frac{dv(t)}{dt}$

$v(t) = L\frac{di(t)}{dt}$

$\Lambda = \int_A \overline{B} \cdot d\hat{a} \ \text{(per turn)} \cdot N$

$w_e(t) = \frac{1}{2}Cv^2(t)$

$w_m(t) = \frac{1}{2}Li^2(t)$

$\tau = RC, \ \tau = \frac{L}{R}$

TEM Sinusoidal Steady State

$\underline{V}(z) = \underline{V}_+ e^{-jkz} + \underline{V}_- e^{+jkz}$

$\underline{I}(z) = Y_o(\underline{V}_+ e^{-jkz} - \underline{V}_- e^{+jkz})$

$k = 2\pi/\lambda = \omega/c = \omega\sqrt{\mu\varepsilon}$

$\underline{Z}(z) = \underline{V}(z)/\underline{I}(z) = Z_o \cdot \underline{Z}_n(z)$

$\underline{\Gamma}(z) = (V_-/V_+)e^{2jkz} = (\underline{Z}_n(z) - 1)/(\underline{Z}_n(z) + 1)$

$\underline{Z}_n(z) = [1 + \underline{\Gamma}(z)]/[1 - \underline{\Gamma}(z)] = R_n + jX_n$

$\underline{Z}(z) = Z_o \cdot (Z_L - jZ_o \tan kz)/(Z_o - jZ_L \tan kz)$

$VSWR = |\underline{V}_{max}|/|\underline{V}_{min}| = \frac{1 + |\Gamma|}{1 - |\Gamma|} = R_{n\,max}$

Electromagnetic Forces

$\overline{f} = q(\overline{E} + \overline{v} \times \mu_o\overline{H}) \ [N]$

$\overline{F} = \overline{I} \times \mu_o\overline{H} \ [N/m]$

$\overline{E}_e = -v \times \mu_o\overline{H}$ (inside conductor)

$vi = \frac{dw}{dt} + f\frac{dz}{dt}$

$f_x = -\frac{dw_e}{dx}\Big|_{Q=const.}$

$f_x = -\frac{dw_m}{dx}\Big|_{\Lambda=const.}$

$\overline{T} = \overline{r} \times \overline{f} \quad P_e = \mu H^2/2, \ \varepsilon E^2/2 \ [N/m^2]$

$T_\theta = -\frac{dw}{d\theta}\Big|_{Q \text{ or } \Lambda = const}$

RLC Resonators

$Z_{series} = R + j\omega L + 1/j\omega C$

$Y_{parallel} = G + j\omega C + 1/j\omega L$

$\omega_o = 1/\sqrt{LC}, \ Q = \frac{\omega_o w_T}{P_{diss}} = \frac{\omega_o}{\Delta\omega}$

EM Resonators

At ω_o, $\langle w_e \rangle = \langle w_m \rangle$

$\langle w_e \rangle = \int_V (\varepsilon|\underline{E}|^2/4) dv$

$\langle w_m \rangle = \int_V (\mu|\underline{H}|^2/4) dv$

$f_{mnp} = \frac{c}{2}\sqrt{(m/a)^2 + (n/b)^2 + (p/d)^2}$

$Q = \frac{\omega_o w_T}{P_{diss}} = \frac{\omega}{2\alpha} = \frac{\omega}{\Delta\omega}, \quad \frac{1}{Q_L} = \frac{1}{Q_E} + \frac{1}{Q_I}$

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