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Department of Electrical Engineering and Computer Science,  
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Quantitative Physiology: Cells and Tissues  
2.791J/2.794J/6.021J/6.521J/BE.370J/BE.470J/HST.541J

**Homework Assignment #1**

Issued: September 8, 2004

Due: September 16, 2004

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**Reading**

- Lecture 2 — Volume 1: 3.1-3.1.3 3.2-3.2.2
  - Lecture 3 — Volume 1: 3.1.4-3.1.5 3.5-3.5.2.1
  - Lecture 4 — Volume 1: 3.6-3.6.1.2 3.7-3.7.2
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**Announcements**

Homework will consist of *Exercises* and *Problems*. Exercises are generally more conceptual and require less number crunching. Exercises often require writing sentences to explain what you have learned. **Written solutions should be submitted for both Exercises and Problems.**

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**Exercise 1.** According to the random walk model, solute molecules move and thereby diffuse because of collisions with water molecules. Solute collisions with other solute molecules are generally ignored under the assumption that the water molecules vastly outnumber the solute molecules. To get a feeling for the validity of this assumption, and to appreciate the number of particles and spatial scales involved, consider the diffusion of potassium ions in the cytoplasm of a red blood cell. Assume that the volume of the cell is 90 fL, and that the concentration of potassium ions in the cytoplasm is 150 mmol/L.

**Part a.** Estimate the number of potassium ions in the cytoplasm of the cell.

**Part b.** Estimate the average distance between potassium ions in the cell.

**Part c.** Estimate the number of water molecules in the cytoplasm of the cell.

**Part d.** Estimate the average distance between water molecules in cytoplasm.

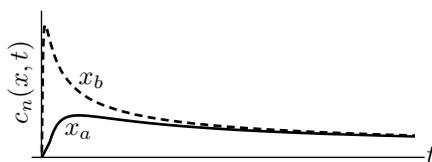
**Part e.** Determine the ratio of water molecules to potassium ions in cytoplasm.

**Exercise 2.** At a junction between two neurons, called a synapse, there is a 20 nm cleft that separates the cell membranes. A chemical transmitter substance is released by one cell (the pre-synaptic cell), diffuses across the cleft, and arrives at the membrane of the other (post-synaptic) cell. Assume that the diffusion coefficient of the chemical transmitter substance is  $D = 5 \times 10^{-6} \text{ cm}^2/\text{s}$ . Make a *rough* estimate of the delay caused by diffusion of the transmitter substance across the cleft. What are the limitations of this estimate? Explain.

**Exercise 3.** The time course of one-dimensional diffusion of a solute from a point source in space and time has the form

$$c_n(x, t) = \frac{n_o}{\sqrt{4\pi Dt}} e^{-x^2/4Dt},$$

where  $n_o$  is the number of moles of solute per unit area placed at  $x = 0$  at  $t = 0$ .  $c_n(x, t)$  is computed as a function of time for locations  $x_a$  and  $x_b$ , and shown in the following figure.



Is  $x_a > x_b$  or is  $x_a < x_b$ ? Explain.

**Problem 1.** A general solution to a first-order linear differential equation with constant coefficients can be written as

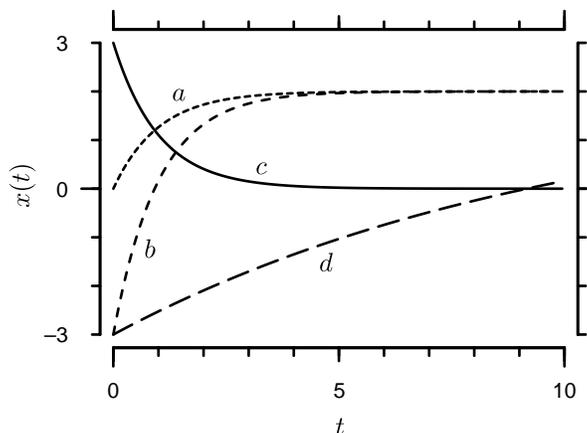
$$n(t) = n_\infty + (n_0 - n_\infty)e^{-t/\tau}.$$

- Determine the slope  $m_0 = dn/dt$  at  $t = 0$  in terms of the constants  $n_\infty$ ,  $n_0$  and  $\tau$ .
- If this slope were extended for  $t > 0$  (i.e., if  $n^*(t) = n_0 + m_0 t$ ), for what value of  $t$  will  $n^*(t) = n_\infty$ ?
- Plot  $n(t)$  and  $n^*(t)$  when  $n_0 = -10$ ,  $n_\infty = 10$ , and  $\tau = 1$ .

**Problem 2.** Four solutions to the differential equation

$$\frac{dx(t)}{dt} + Ax(t) + B = 0$$

are shown in the following plot.



**Part a.** Find values of  $A$  and  $B$  that are consistent with curve  $a$ . Are these values unique? If not, find a second set of constants that are consistent.

**Part b.** Repeat part a for curve  $b$ . Compare these results to those of part a. Explain similarities and differences.

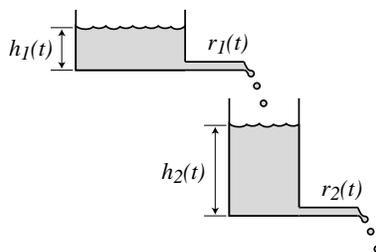
**Part c.** Repeat part a for curve  $c$ . Compare these results to those of parts a and b. Explain similarities and differences.

**Part d.** Repeat part a for curve  $d$ . Determine all possible values of  $A$  and  $B$  for this case.

**Problem 3.** Assume that  $n_o$  mol/cm<sup>2</sup> of sucrose (with a diffusion constant  $D \approx 0.5 \times 10^{-5}$  cm<sup>2</sup>/s) are placed in a trough of water at a point  $x = 0$  at time  $t = 0$ . Assume that the concentration of sucrose is a function of  $x$  and  $t$  only.

- Show that for any fixed point  $x_p$  the maximum concentration occurs at time  $t_m = x_p^2/2D$ .
- How long does it take for the concentration to reach a maximum at  $x = \pm 1$  cm?

**Problem 4.** The following figure illustrates a cascaded system of two water tanks. Water flows out of the first tank and into the second at a rate  $r_1(t)$ , and out of the second tank at a rate  $r_2(t)$ .

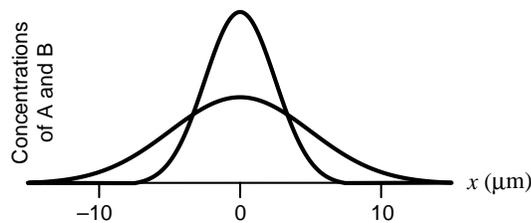


The rates of flow out of the tanks are proportional to the heights of the water in the tanks:  $r_1(t) = k_1 h_1(t)$  and  $r_2(t) = k_2 h_2(t)$ , where  $k_1$  and  $k_2$  are each 0.02 m<sup>2</sup>/minute. The height of tank 1 is 1 m and that of tank 2 is 2 m. The cross-sectional area of tank 1 is  $A_1 = 4$  m<sup>2</sup> and that of the second tank is  $A_2 = 2$  m<sup>2</sup>. At time  $t = 0$ , tank 1 is full and tank 2 is empty.

- If the height of water in tank 2 ever exceeds the height of the tank (2 m), the water will overflow. Will the water ever overflow? Explain.
- Set up a system of differential equations to determine  $h_2(t)$ . Solve the equations to determine an expression for  $h_2(t)$ .
- At what time does the water in tank 2 reach its peak? What will be the maximum height of water ever achieved in tank 2?
- At what time will the water stop flowing out of tank 1? Explain your answer in mathematical terms and then in physical terms.
- If both tanks were full at  $t = 0$ , would the second tank ever overflow? Explain.

**Problem 5.** Transport by diffusion tends to move solutes from regions of high concentration to regions of low concentration, thereby making the spatial distribution of solute more uniform. Thus diffusion is often associated with mixing. However, diffusion can also be used to separate solutes in a mixture. Such separation is important as a mechanism to purify mixtures of biological materials (such as mixtures of DNA fragments) and is currently being exploited in microfabricated systems for biological and medical analysis (which we will look at in greater detail in the Microfluidics Laboratory Project).

Assume that a mixture of two molecular species is loaded into a long and narrow channel filled with water. Let  $x$  represent distance in the longitudinal direction and assume that  $n_0$  molecules of solute A and  $n_0$  molecules of solute B are loaded into location  $x = 0$  at time  $t = 0$ . Assume that solutes A and B have different molecular weights and that their diffusivities are  $D_A = 10^{-7} \text{ cm}^2/\text{s}$  and  $D_B = 4 \times 10^{-7} \text{ cm}^2/\text{s}$ , respectively. The following figure illustrates how the two species tend to separate as they diffuse.



**Part a.** As time elapses, the number of molecules of solute A that remain in a test region within the  $10\mu\text{m}$  of the starting position (i.e.,  $-10\mu\text{m} < x < 10\mu\text{m}$ ) decreases. Determine the amount of time  $t_0$  required to get 5% of solute A to leave the test region.

**Part b.** At time  $t = t_0$  found in part a, determine the amount of solute B in the test region.

**Part c.** Determine the ratio of solute A to solute B in the test region at time  $t = t_0$ . Explain how diffusion has led to a separation of solute A from solute B.