MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Spring 2006)

Recitation 12 Solutions

April 5, 2006

1. (a) First note that $\hat{X}$ should be a r.v., not a number. In particular, we are to minimize over all r.v.'s $\hat{X}$ that can be expressed as functions of Y. From lecture, $\hat{X} = \mathbf{E}[X|Y]$.

   Now, take conditional expectations, to get $Y = \mathbf{E}[Y|Y] = \mathbf{E}[X|Y] + \mathbf{E}[W|Y]$. Since there is complete symmetry between X and W, we also have $\mathbf{E}[X|Y] = \mathbf{E}[W|Y]$, which finally yields $\mathbf{E}[X|Y] = Y/2$.

   (b) In the dependent case, we cannot simply conclude that the distribution $f_{X,W}(x,w)$ is symmetric in its two argument (i.e., $f_{X,W}(x,w) = f_{X,W}(w,x)$), even though the marginals $f_X(x), f_W(w)$ are the same.

   Since $f_{X,W}(x,w)$ is not symmetric, $\mathbf{E}[X|Y] \neq \mathbf{E}[W|Y]$ in general.

   So in this case, one cannot really solve the problem with the available information, we really need the joint distribution in order to compute the conditional expectations.

   The solution given in the independent case still works, though, for any symmetric distribution.

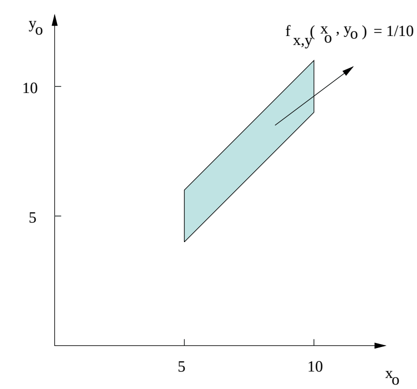
2. (a) The minimum mean squared error estimator $g(Y)$ is known to be $g(Y) = \mathbf{E}[X|Y]$. Let us first find $f_{X,Y}(x,y)$. Since $Y = X + W$, we can write

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{2} & \text{if } x-1 \leq y \leq x+1 \\ 0 & \text{otherwise} \end{cases}$$

   and, therefore,

$$f_{X,Y}(x,y) = f_{Y|X}(y|x) \cdot f_X(x) = \begin{cases} \frac{1}{10} & \text{if } x-1 \leq y \leq x+1 \text{ and } 5 \leq x \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

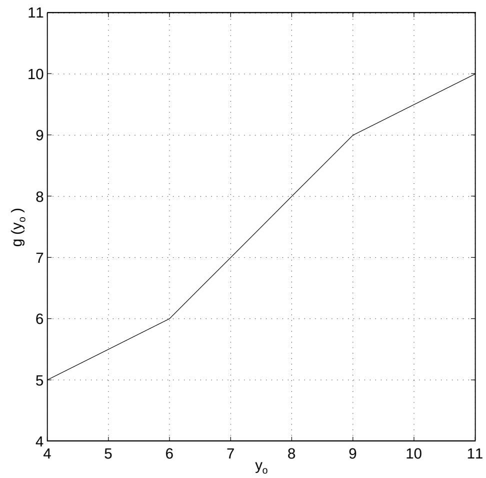
   as shown in the plot below.



   We now compute $\mathbf{E}[X|Y]$ by first determining $f_{X|Y}(x|y)$. This can be done by looking at the horizontal line crossing the compound PDF. Since $f_{X,Y}(x,y)$ is uniformly distributed in the defined region, $f_{X|Y}(x|y)$ is uniformly distributed as well. Therefore,

$$g(y) = \mathbf{E}[X|Y=y] = \begin{cases} \frac{5+(y+1)}{2} & \text{if } 4 \leq y < 6 \\ y & \text{if } 6 \leq y \leq 9 \\ \frac{10+(y-1)}{2} & \text{if } 9 < y \leq 11 \end{cases}$$

   The plot of $g(y)$ is shown here.



(b) The linear least squares estimator has the form

$$g_L(Y) = \mathbf{E}[X] + \frac{\text{cov}(X,Y)}{\sigma_Y^2}(Y - \mathbf{E}[Y])$$

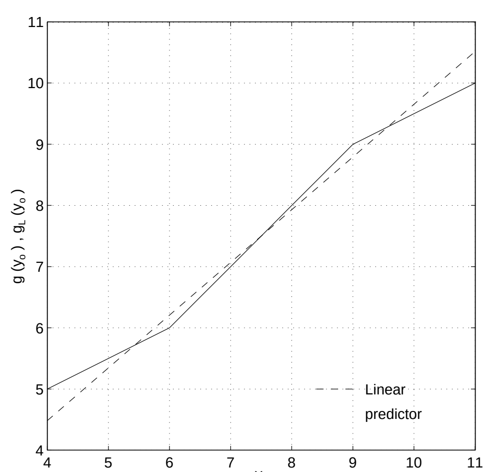
where $\text{cov}(X,Y) = \mathbf{E}[(X - \mathbf{E}[X])(Y - \mathbf{E}[Y])]$. We compute $\mathbf{E}[X] = 7.5$, $\mathbf{E}[Y] = \mathbf{E}[X] + \mathbf{E}[W] = 7.5$, $\sigma_X^2 = (10-5)^2/12 = 25/12$, $\sigma_W^2 = (1-(-1))^2/12 = 4/12$ and, using the fact that X and W are independent, $\sigma_Y^2 = \sigma_X^2 + \sigma_W^2 = 29/12$. Furthermore,

$$\begin{aligned}
\text{cov}(X,Y) &= \mathbf{E}[(X - \mathbf{E}[X])(Y - \mathbf{E}[Y])] = \mathbf{E}[(X - \mathbf{E}[X])(X - \mathbf{E}[X] + W - \mathbf{E}[W])] \\
&= \mathbf{E}[(X - \mathbf{E}[X])(X - \mathbf{E}[X])] + \mathbf{E}[(X - \mathbf{E}[X])(W - \mathbf{E}[W])] \\
&= \sigma_X^2 + \mathbf{E}[(X - \mathbf{E}[X])]\mathbf{E}[(W - \mathbf{E}[W])] = \sigma_X^2 = 25/12.
\end{aligned}$$

Note that we use the fact that $(X - \mathbf{E}[X])$ and $(W - \mathbf{E}[W])$ are independent and $\mathbf{E}[(X - \mathbf{E}[X])] = 0 = \mathbf{E}[(W - \mathbf{E}[W])]$. Therefore,

$$g_L(Y) = 7.5 + \frac{25}{29}(Y - 7.5).$$

The linear estimator $g_L(Y)$ is compared with $g(Y)$ in the following figure. Note that $g(Y)$ is piecewise linear in this problem.



3. The problem asks us to find $P(x_1^2 + x_2^2 \leq \alpha)$. The information given completely determines the joint density function, so we need only to perform the integration:

$$\begin{aligned}
P(x_1^2 + x_2^2 \leq \alpha) &= \iint_{x_1^2+x_2^2 \leq \alpha} \frac{1}{2\pi} e^{-\frac{x_1^2+x_2^2}{2}} dx_1 dx_2 \\
&= \int_0^{2\pi} \int_0^{\alpha} \frac{1}{2\pi} e^{-\frac{r^2}{2}} r dr d\theta \\
&= 1 - e^{-\frac{\alpha^2}{2}}
\end{aligned}$$