

## 6.041SC Probabilistic Systems Analysis and Applied Probability, Fall 2013 Transcript – Recitation: A Chess Tournament Problem

Hi. Welcome back. Today, we're going to do a fun problem called the chess tournament problem. Now, it's a very long problem, so I just want to jump straight in.

Essentially, the problem statement describes a very special chess tournament, which involves players named Al, Bo, and Chi. Now Al is the current reigning championship, and Bo and Chi are this year's contenders, and, of course, they're vying with each other to beat out Al and become the new champion. And so essentially, the tournament is divided into two rounds-- a first round, during which Bo and Chi play against each other, and then a second round, during which the surviving contender from the first round plays against Al. And the problem statement also gives you a bunch of information like what's the probability that Bo beats Chi in a particular game, et cetera.

So without further ado, let's get started on part a. In part a, the first thing we're asked to compute is the probability that a second round is required. Now, to save myself some writing, I used the notation  $R_2$  to denote that event. So we are interested in probability of  $R_2$ .

Now, I claim that this problem is very sequential in nature so I would like to draw a tree to describe what's happening. So in the first part of the tournament, when Bo and Chi play their first game, exactly one of two things can happen-- either Bo can win or Chi can win. And we're told by the problem statement that Bo wins with the probability of 0.6 and, therefore, Chi must win with the probability of 0.4, right? Because these two possibilities must sum to 1, because either this must happen or this happen.

Now, let's imagine that the first game has been played and that Bo won. Well, during the second game, there's still two options for the outcome-- Bo could win the second game or Chi could win the second game. And because the problem statement says that in every scenario Bo always wins against Chi with the probability of 0.6, we can go ahead and put a 0.6 along this branch as well. Similarly, 0.4 here. And similar logic, you've got a tree that looks like this.

And for those of you who haven't seen trees before, it's just a structure that looks something like this. And it helps us do better accounting. It helps us keep straight in our head what are the various outcomes, so that we don't get confused. And so very quickly here, you can see that there's four possible outcomes.

So each node in this tree corresponds to an outcome. And the leaves are those nodes at the furthest stage. And it's convention to draw the probability of a particular-- so for instance, the probability that Bo wins the first game-- it's just convention to draw that probability over the corresponding branch. And the reason why such diagrams are so useful is because to compute the probability of a particular outcome, if you've designed your tree correctly, all you have to do is multiply the probabilities along the branches that get into that outcome.

So let's see that in action. When is a second round required? Well, a second round is required here, right? Because in this case, Bo would be the surviving challenger and he'd play the next round against Al.

It's also required here. But of course, it's not required here or here, because no second round is played. And so these two outcomes comprise the event R2.

And now, to get the probability of this outcome, you multiply along the branches. So 0.6 times 0.6 give you 0.36. And 0.4 times 0.4 gives you 0.16. And we're almost done.

We know that these two events are disjoint, because if Bo won the first two games, then, certainly, Chi could've won the first two games. And so you can just sum the probabilities to get the probability of the reunion. So the probability of R2 is equal to the probability that Bo won the first two games or Chi won the first two games. And that's equal to 0.36 plus 0.16, which is equal to a 0.52.

OK, now the second part of part a asks for the probability that Bo wins the first round. And this is first round. This is a very straightforward one. So Bo wins the first round, that correspondence only to this particular outcome. And we already know the probability associated with that outcome is equal to the 0.36. So we're done with that one.

And now the last part is sort of an interesting one. It asks for the probability that Al retains his championship this year. So I'm going to just call that A for short. A is the event that Al retains his championship this year. And for that we're going to need a larger tree, because Al has a lot of activity in the second round, and so far our tree only describes what happens in the first round.

Now, to save time, I've actually drawn the rest of the tree over there up in the corner. So let's get rid of this one and let's look at the full tree. So let's see when does Al retain his championship? Well, Al certainly retains his championship here, right? Because no second round is required. Similarly, here.

Al retains his championship here, because the second round was required, but Al beat Bo. And similarly, here Bo didn't win both games in the second round against Al, so Al wins. Here, Bo is the new championship. So we don't want to include about one. And sort of by symmetry, we also get this one and this one.

So by my argument before, we know that the outcomes that comprise our event of interest are this one, this one, this one, this one, this one, and this one. So we could multiply the probabilities along each branch and sum them, because they're disjoint, to get the total probability. But we're not going to do that because that's a lot of algebra. Instead, we're going to look at the complement of the event.

So we're going to notice, there's only two branches on which Al does not retain his current championship. So P of A is, of course, equal to 1 minus P of A. And we're going to get P of A by inspection. I'm sorry, so P of A compliment. I'm just testing you, guys.

So P of A compliment corresponds to here and to here, because those are the outcomes where A1 didn't win. And so again, you multiply along the branches to get the probabilities. So you get 0.6 squared times 0.5 squared plus 0.4 squared times 0.3 squared. And if you do all the algebra, you should get around 0.8956.

So we're cruising through this problem. Let's go to part b. Part b is a little bit less straightforward than part a, because it starts asking you for conditional probabilities, as opposed to a priori probabilities. So in the first part-- and again, I'm going to continue my notation with R2-- we want the probability that Bo is the surviving challenger-- so I'm just going to use B to denote that-- given R2.

Now, by definition, you should remember from lecture that this is equal to probability of B and R2 divided by the probability of R2. And of course, we've already computed this value right up here with part a. We know it's 2.5.

So we don't have to do any more work there. We only have to look at the numerator. So we need to go and figure out what nodes in that tree correspond to the event B intersect R2.

So let's use a new color. Let's see, Bo is the surviving challenger here only, right? And R2 is automatically satisfied, right? Because a second round is required there and there, not on those two. But here Chi is the surviving challenger, not Bo, so we're really only interested in that node. And you multiply along the branches to get the probabilities. So we have 0.36 over 0.52, which is approximately equal to 0.6923.

OK, now, the next part wants the conditional probability that AL retains his championship, conditioned, again, on R2. So we already have A being the event A1 retains his championship. So we want the probability of A, given R2.

And let's just apply the direct definition of conditional probability again. You get P of A and R2 divided by a probability of R2. Of course, we have the probability of R2 already, so we just need to find the node in the tree that corresponds to A and R2.

So where is R2? R2 is going to correspond to every node to the right that is not one of these two. So a second round is required here, here, here, here, here, and here.

Now, where does A1 retain his championship? So A1 retains his championship here. He retains his championship here. He retains his championship here and here, but no second round is required, so these guys don't belong in the intersection. But this does, and this does. So we can again multiply the probabilities along the branches and then some them.

So let's see, we get-- this marker's not working very well, so I'm going to switch back to the pink-- so you get 0.6 squared times 0.5. That gets rid of this one. And then we want 0.6 squared times 0.5 squared. That gets rid of that one.

And then plus-- let's see-- 0.4 squared times 0.7, which takes care of this one. And then lastly, 0.4 squared times 0.3 times 0.7. And that is a long expression. But it happens to be about 0.7992.

OK, so we are done with part b and we can move along to part c. And I am, since we're running out of room, I'm actually just going to erase this. And hopefully you guys have had a chance to copy it down by now. If not, you can always pause the video and go back.

So let's see, part c asks us given that the second round is required and that it comprised of one game only. So let's denote  $I$ . So let's  $I$  be the event that the second round was one game only. So essentially, in math conditioned on  $R_2$  and  $I$ , what is the probability that it was  $B_0$  who won the first round?

So let's let  $B$  be the event that  $B_0$  won the first round. OK, so again translating the English to math, we just want the probability of  $B$  given  $R_2$  and  $I$ . Now, I am once again going to use the definition of conditional probability.

You might be concerned that we haven't defined explicitly yet the definition of conditional probability, when what lies behind the conditioning bar is not a single event, but it's rather an intersection of an event. And so my claim to you is that it doesn't matter and that the same exact definition applies. But we'll go through it slowly.

So  $R_2$  is an event,  $I$  is an event, and we know that the intersection of two events is itself an event. So I'm going to make up a new letter, and I'm going to call this event  $W$ . So just using the new notation, this is equal to probability of  $B$ , given  $W$ .

Now, this is the normal definition that we know. We know that this is probability of  $B$  intersect  $W$  over probability of  $W$ . And then we just resubstitute what the definition of  $W$  was. And so if you do that over here, you get probability of  $B$  and  $R_2$  and  $I$  divided by probability of  $R_2$  and  $I$ .

So hopefully, jumping from here ahead to here, you see that the definitions act exactly the same way. But these are two very short intermediate steps that should help you convince yourself that the same definition still works. So let's start with the denominator, because the denominator looks a little bit easier. Where is  $R_2$  and  $I$  in our tree?

Well, let's see. Here, a second round was required, but it comprised two games. Same with this one. Here, a second round was required and it was comprised only of one game. So this is good. This is one of the outcomes that we're looking for.

Here, no second round was required. So this doesn't qualify. Same with this one. Here, a second round was required, and there was only one game, so that's good. And then these don't qualify for the same reasons as we set up there.

So we just have to multiply the probabilities along those branches. And we see that it's  $0.4$  squared times  $0.7$  plus  $0.6$  squared times  $0.5$ .

OK, we're almost done. We just need to look at the intersection of  $R_2$  and  $I$ . So  $R_2$  and  $I$  are the ones we've already circled. But now, we want to add one more constraint, which is that  $B_0$  had to have won the first round.

And so we see here that Chi won the first round, if we're looking at this outcome. And so he's no good. Let's use a different color. Let's see, maybe this one. But here Bo did win the first round.

So we're going to get  $0.6^2 \times 0.5$ . And I got that, of course, just by multiplying the probabilities along the right branches. And this, if you're curious, comes out to be about 0.6164.

OK, so I know that was a lengthy problem, but you should feel really comfortable now doing sort of basic probability manipulations. One thing that this problem emphasized a lot was your ability to compute conditional probabilities. So you saw me apply the definition of conditional probability twice in part b. And then you saw me apply the definition again in part c in a sort of slightly modified way. So that's one thing that you should have gotten out of this problem.

And then another thing is that hopefully, you noticed that by using a tree diagram, we made the problem much easier. We almost didn't even have to think about computing probabilities anymore. We reduced the problem to just saying, OK, what are the outcomes that comprise our event of interest? And then once you select those, to compute their probability you multiply the probabilities along the branches.

You have the right to just add those together, because if you draw your tree correctly, all of these guys should be disjoint from one another. So you have to be careful, of course, to set up your tree appropriately. But once you do set up your tree appropriately, your life is much simpler. So that's it for today. And we'll see you next time.

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6.041SC Probabilistic Systems Analysis and Applied Probability  
Fall 2013

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