

6.041SC Probabilistic Systems Analysis and Applied Probability, Fall 2013 Transcript – Recitation: Competing Exponentials

Hi, in this problem, we're going to look at competing exponential. So we have three exponential random variables, X with parameter λ , Y with parameters μ , and Z with parameter ν . And we want to calculate some probability. And the probability that we want to calculate is the probability that X is less than Y is less than Z .

Now we can reinterpret this as 3 plus Poisson processes. Because the link between exponentials and Poisson processes is that the inter-arrival time of Poisson processes are exponentially distributed. So you can think of X being the time until the first arrival in a Poisson process with parameter λ . And same thing for Y is the first arrival time of a Poisson process with parameter μ . The same thing for Z and ν .

And so in that interpretation, X less than Y less than Z , you could interpret as a race, meaning that X finishes first followed by Y and then Z comes in last. So with that interpretation, let's see if we can calculate what this probability is. We can rewrite this probability as a combination of two things occurring. One is that X is less than the minimum of Y and Z . And then the other is that Y is less than Z .

So what is this first event mean? This first event means that X comes in first. And it doesn't matter whether Y comes in second or Z comes in second. So we first say that X has to come in first which means it has to beat the better of Y and Z . And then that combined with the fact that Y does better than Z is the same thing as saying that X is first, Y is second, and Z is third.

And now, let's try to argue using Poisson processes, that these two events are actually independent. So this event occurring means that X is smaller than Y and Z . So let's take these Poisson processes, and because these random variables are assumed to be independent, these are independent Poisson processes. So we can merge them. So let's merge these two. And we'll get a Poisson process that has rate μ plus ν .

And we can also merge this first one and that one. And we'll get another Poisson process with parameter λ plus μ plus ν . So in that context, what does it mean that X is less than the minimum Y and Z ? It just means that in this merged process, the first arrival came from the X process. In that case, if that's true, then X is less than minimum Y and Z . Well, let's say that event does occur that the first arrival is from the X process. Now we're interested in what the order of the second two arrivals are. Is it Y first and then Z ? Or Z first and then Y ?

Well, it doesn't matter because of the fresh start property. Because after this arrival comes, and say it is from the X process, the Poisson processes start anew, and they're still independent. And so what happens after that is independent of what happened here, when X arrived. And so whether Y came first followed by Z , or Z came first followed by Y is independent of what happened here. And so because of that, these two events are independent, and so when we have

the probability of the intersection of two independent events, we can write that as the product of those two probabilities.

Now, what is the probability of this first event? The probability that X is less than the minimum Y and Z? Well, we just said that that corresponds to the first arrival of this merge process being from the X process. Well, that probability is λ over $\lambda + \mu + \nu$. So it's equal to this ratio where the process that you're interested in, its rate comes in the numerator. And then the merged rate is on the denominator,

And what about the second one? What's the probability that Y is less than Z? Well, let's go now to this merge process where we merged just the Y and Z processes and see which one comes first. Well, in that case what we want to know is in this merge process, what is the probability that the first arrival came from the Y process? Well, analogously, that probability is going to μ over $\mu + \nu$. And that gives us our answer.

And so we see that what looked like a pretty complex calculation when we reinterpreted it in terms of Poisson processes, it becomes relatively easy to solve. But this still seems like a complicated expression. So let's try to check to see whether it actually makes sense. So one way to do that is to look at a specific example of the choice of λ , μ , and ν , and see if it actually makes sense.

So one example is suppose that all three of these parameters are the same. Well, if they're all the same then this probability, the first part becomes 1 over 3, $1/3$. And the second one is 1 over 2. And so if all three parameters are the same, probability becomes $1/6$. And let's see if that makes sense. If all three parameters are the same, that means that these rates, these arrival rates are all the same. And what that means is that any three ordering of these three arrivals is as likely as any other ordering. And what we're interested in is the probability of one's particular ordering happening which is X first then Y then Z.

But if everything is symmetric then any of the orderings is as likely as any other one. And how many orderings are there? Well, there's three choices for who comes in first. Two for who comes in second. And one for who comes in last. So there's a total of six possible orders in which these three contestants, if you think of it that way, could finish this race. And out of none of those, we want the probability that one of those outcomes happens. And so the probability should be $1/6$. And that's what our formula tells us. So as I said, in this problem, we saw how to reinterpret exponentials in the context of Poisson processes that helped us solve a--

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