PROFESSOR:

Today we're going to start with a simple problem that many of you may have already encountered. For example, when you got a student loan or if your family took a loan out on your house, you know, a home mortgage loan, and it's the problem of pricing an annuity. An annuity is a financial instrument that pays a fixed amount of money every year for some number of years, and it has a value associated with it.

For example, with a student loan, the value is the amount of money they gave you to pay MIT, then later in life-- every year or every month-- you're going to send them a check. And you want to sort of equate those two things to find out, are you getting enough money for the money you're going to pay back monthly sometime in the future?

So let's define this, and there's a lot of variations on annuities, but we'll start with one that's called an n-year $m-dollar payment annuity, and it works by paying m dollars at the start of each year, and it lasts for n years. Now, usually n is finite, but not always, and in a few minutes we'll talk about the case when it's infinite.

But this includes home mortgages, where you pay every month for 30 years. Megabucks, the lottery. They don't actually give you the million dollars. They give you $50,000 per year for 20 years, and call it a million bucks. Retirement plans. You pay in every year, and then you get some big lump sum later. Life insurance benefits, you know, and so forth.

Now, if you go to Wall Street, this is a big deal. A lot of the stuff that happens on Wall Street involves annuities in one form or another, packaging them all up and, in fact, we'll look at later when we do probability. It was how these things were packaged and sold that led to the sub-prime mortgage disaster, and we'll see how some confusion over independents, when you look at random variables, led to the global recession, a real disaster.

Of course, some people understood how all that worked, made hundreds of billions of dollars at the same time. So it was sort of money went from one place to another. So it's pretty important to understand how much this is worth. What is this instrument-- a piece of paper that says it will pay you m dollars at the beginning of each year for n years, what is that worth today?

For example, say I gave you a choice-- the Megabucks choice-- $50,000 a year for 20 years or a million dollars today. How many people would prefer the $50,000 a year for 20 years? A few of you. How many would prefer the million bucks right up front? Much better. OK. Always better to have the cash in hand, because there's things like inflation-- pretty low now-- interest. You can put the money in the bank or invest it and make some money hopefully.

So the million dollars today is a lot better, which is why the State pays you 50 grand a year for 20 years. It's better for them, and they call it a million bucks. So that was pretty clear, but what if I gave you this option-- 700 grand today or 50 grand a year for 20 years? How many people want the cash upfront-- 700 grand only. A few.

How many people want 50 grand a year for 20 years? All right, we're almost-- that's pretty close to half way. How about 500 grand today versus 50 grand a year for 20 years? How many want half a million today? A lot of people like the cash. You know, it's this kind of a time, you have the recession, it's a disaster on Wall Street. You know, Street Wall Street didn't like the cash.

How many want, instead of a half a million up front, 50 grand a year for 20 years? All right, now we're about halfway. All right, well that's pretty good. So we're going to find out what you should pay, or at least one way of estimating that.

Now, to do that, we've got to figure out what $1 today is worth in a year. And to do that, we make an assumption, and the assumption is that there's a fixed-- we'll call it an interest rate. It's sort of the devaluation of the money per year. And we're going to call it p. Later we'll plug in values for p, but you can think of it as like 6%, 1%. You know, it's the money, if you put money in a bank, they'll give you some percent back every year. And, of course, the fact that different people have different ideas of what this would be, allows people to make money on Wall Street. As we'll see, a slight difference in p can make big differences in what the annuity is worth.

So for example, $1 today is going to equal 1 plus p dollars in one year. Similarly, $1 today-- how much is that going to be worth in two years? Say that p stays fixed, the same over all time. One plus p squared, because every year you multiply what you got by 1 plus p, because that's the interest you're getting. All right, we'll think of it in terms of interest. In three years, $1 today is worth 1 plus p cubed in three years and so forth.

All right. Now, what we really care about is what's $1, or m dollars, worth today if you're getting it next year? So we need to sort of flip this back the other way. So what is $1 in a year worth today in terms of p? So if I'm going to be paid $1 in a year, what would be the equivalent amount to be paid today? One over 1 plus p, because what's happening here is, as you go forward in a year, you just multiply by 1 plus p. So 1 over 1 plus p turns into $1 in a year-- being paid in a year. All right. What is $1 a year in two years worth today? One over 1 plus p squared. So $1 in two years is worth this much today.

Well, now we can use this to go figure out the current value of that annuity. We just figure out what every payment is worth today and then add it up. So we'll put the payments over here, and we'll compute the current value of every payment on this side. So with the annuity, the way we've set it up is it pays n dollars at the start of every year, so the first payment is now. So the first of the n payments is now, and since it's being paid now, that's worth m dollars. There's no devaluation.

The next payment is m dollars in one year, and so that's going to be worth m over 1 plus p today. And the next payment is m dollars in two years. That's worth m over 1 plus p squared, and we keep on going until the last payment. It's the n-th payment, so it's m dollars in n minus 1 years. And so that's going to be worth m over 1 plus p to the n minus 1.

All right. So we can compute the current value of all those payments, then the annuity is computed-- the value is computed just by adding these up, of all the current values. So the total current value is the sum i equals 0 to n minus 1 of m over 1 plus p to the i. And that is the total current value. That's what you should pay today for the annuity. Any questions? What we did here?

All right. Well, of course, what we'd like is a closed form expression here. Something that's simple so we could actually get a feel without having to add up all those terms, and that's not hard to get. In fact, let's put this sum in a form that might be more familiar.

This equals-- we'll pull the m out in front-- and let's use x to be 1 over 1 plus p to the i. And so x equals 1 over 1 plus p, and I wrote it this way because this might be familiar. Does everybody remember that from-- I think it was the second recitation? Anybody remember the formula? What this evaluates to? The sum of x to the i, where i goes from 0 to n minus 1?

Remember that? One minus x to the n. Remember 1 minus x. In the second recitation, I think, we proved that this equals that. What was the proof technique we used? Induction. OK?

So, in fact, there's a theorem here. For all n bigger and equal to 1 and x not equal to 1, we proved the sum from i equals 0 to n minus 1 x to the i equals 1 minus x to the n over 1 minus x. And so this is a nice, closed form. No sum any more, just that, which is nice.

Now, induction proved it was the right answer. Once you knew it-- we gave it to you-- using induction to prove that theorem wasn't hard. What we're going to look at doing this week and next week is figuring out how to figure out this was the answer in the first place. Methods for doing that-- to evaluate the sum-- and there's a lot of ways that you can do that particular sum.

Probably the easiest is known as the perturbation method. This sometimes works. Certainly with sums like that, it often works. The idea is as follows. We're trying to compute the sum S, which is 1 plus x plus x squared plus x to the n minus 1, and what we're going to do is perturb it a little bit and then subtract to get big cancellation.

In this case, it's pretty simple. We multiply the sum by x to get x plus x squared plus-- I've defined S to be that, x times S-- well, I get x plus x squared and so forth, up to x to the n, and now I can subtract one from the other and almost everything cancels. So I get 1 minus x times S equals 1. These cancel, cancel, cancel. Minus x to the n. And therefore S equals 1 minus x to the n over 1 minus x.

So that's a vague method. This gets used all the time in applied mathematics, and they call it the perturbation method. Take your sum, wiggle it around a little bit, get something that looks close, subtract it, everything cancels, life is nice, and all of a sudden you've figured out the answer.

So getting back to our annuity problem, we can plug that formula back in here. So the value of the annuity is m times 1 minus x to the n over 1 minus x. We'll plug in x equals 1 over 1 plus p, and we get m 1 minus 1 over 1 plus p to the n over 1 minus 1 over 1 plus p, just plugging in. And now to simplify this, I'll multiply the top and bottom by 1 plus p, and I'll get 1 plus p minus 1 on the bottom. Just gives me a p on the bottom. I have a 1 plus p on the top minus 1 over 1 plus p to the n minus 1, all over p.

So now we have a formula-- closed form expression formula-- for the value of the annuity. All's we've got to plug in is m, the payment every year, n, the number of years, and then p, the interest rate. And so, for example, if we made m be $50,000, as in the lottery. We made n be 20 years, and say we took 6% interest, which is actually very good these days, and I plug those in here, the value is going to be $607,906.

All right. So those of you that preferred 700 grand-- if you assume 6% interest-- you're right. Those of you who preferred 500 grand, no, you're better off waiting and getting your 50 grand a year.

Now, of course, if the interest rate is lower, well, that changes things. That shifts it even more. The annuity is worth even more if the interest rate is lower-- if p is smaller. In fact, say p was 0. Say the interest rate is 0, so $1 today equals $1 tomorrow, then what is the lottery worth? A million dollars. And the bigger p gets, the less your payment is worth. Any questions about that? OK.

What if you were paid $50,000 a year forever-- you live forever or it goes to your estate and your heirs, $50,000 a year forever or a million dollars today. How many people want the million dollars today? How many want 50 grand a year forever? Sounds good. You know, that's an infinite amount of money, sort of. It's not as good as it sounds. Let's see why.

So this is a case where n equals infinity, and so I'll claim that if n equals infinity, then the value of this annuity is just m times 1 plus p over p. Let's see why that's the case. You know, it sounds hard to evaluate, because it's an infinite number of payments, but what happens here when n goes to infinity? What happens to this thing? That goes to 0 as n goes to infinity, as long as p is bigger than 0.

So we're going to assume 6% interest. So that goes away, so the annuity is worth just that, m times 1 plus p over p, because the limit as n goes to infinity of 1 over 1 plus p to the n minus 1, that's going to 0. So the value for m is $50,000, and at 6% V is only $883,000. So you're better off taking a million dollars today than $50,000 a year forever.

Now, if you think about it, and think about it as an interest rate, why should it be obvious that you're better off with a million dollars today than 50 grand a year forever? Think about what you could do with that million dollars if you had it today at 6% interest. What would you do with it to make more money than 50 grand a year forever? Yeah.

AUDIENCE: You could have like-- you could make $50,000 per year just off of [INAUDIBLE].

PROFESSOR: Yeah. In this model, if the interest rate is 6%, you can put in the bank. It makes 6% every year, that's 60 grand a year forever. Better than 50 grand a year forever. So maybe it's-- even without doing the math, you can tell which way it's going to go, but this tells you exactly what it's worth. Any questions? OK. Yeah.

AUDIENCE: [INAUDIBLE] current value-- that's how much it's worth to you right now.

PROFESSOR: Yeah.

AUDIENCE: So and, like, if you have $1 in the future--

PROFESSOR: Yeah.

AUDIENCE: --where S right now is m over 1 plus p--

PROFESSOR: Yeah.

AUDIENCE: --is it worth less to you [INAUDIBLE] then later on, it's going to be worth more to you.

PROFESSOR: Ah, so if you move yourself forward in time, $1 a year, in a year it'll worth $1 to you--

AUDIENCE: Yeah.

PROFESSOR: --but today it's worth less than $1 to you, because you could take the dollar today and invest in the bank, and it's worth more in a year, because the money grows in value, as a way to think of it, because you can earn interest on it. What's that?

AUDIENCE: You could spend it.

PROFESSOR: Yeah. Yeah, if you just spend it, well, then at least you had the use of what you spent it on for the year. So there's some other kind of value, right? Maybe you bought a house or something that-- maybe something that even appreciated in value. OK.

But these things get sort of squishy, and that is where a lot of people make money on Wall Street, is because different companies have different needs for money. They have different views of what the interest rates are going to be, and you can play in the middle and make a lot of money that way.

So more generally, there's a corollary to the theorem, and that is that if the absolute value of x is less than 1, then the sum i equals 0 to infinity x to the i is just 1 over 1 minus x. We didn't prove this back in the second recitation, because there's no n to induct on here, but the proof is simple from the theorem. And it's simply because if x is less than 1, an absolute value, as n goes to infinity, that goes to 0, and so you're just left with 1 over 1 minus x.

So, for example, what's this sum? This one you all know, I'm sure. Out to infinity. What's that sum to? To 2. Yeah. It's 1 over 1 minus 1/2, which is 2. What about this sum out to infinity? What does that sum to?

AUDIENCE: [INAUDIBLE]

PROFESSOR: Yeah, 3/2, 1 over 1 minus 1/3 is 3/2. So, easy corollaries. These are all examples of geometric series. That's what a definition of a geometric series is. Something that's going down by a fixed-- each term goes down by the same fixed amount every time. And geometric series, generally sum to something that is very close to the largest term. In this case, it's 1. Very common, because of that formula. It's 1 over 1 minus x. Any questions about this or geometric series? All right.

Well, those are straight geometric sums. Sometimes you run into things that are a little bit more complicated. For example, say I have this kind of a sum, i equals 1 to n, i times x to the i. Now, those are adding up x plus 2x squared plus 3x cubed, and so forth, up to n x to the n. You know, that's a little more complicated. The terms are getting-- decreasing by a factor of x, increasing by 1 in terms of the coefficient every time. A little trickier.

So say we wanted to get a closed form expression for that? There are several ways we can do it. The first would be to try to use perturbation-- the perturbation method. Let's try that. So we write S equals x plus 2x squared plus 3x cubed plus nx to the n, and let's try the same perturbation. Multiply by x, I get that x squared plus 2x cubed plus n minus 1x to the n, plus nx to the n plus 1. And then I subtract to try to get all the cancellation.

So then I do that. I get 1 minus x times S. Well, I didn't quite cancel everything. X plus x squared plus x cubed plus x to the n plus n-- or minus nx to the n plus 1. Ah, it didn't quite work. Anybody see a way that I can fix this up? What about this piece? That's still a mess here. Can I simplify that? Yeah?

AUDIENCE: [INAUDIBLE]

PROFESSOR: Yeah. There's

a simpler way. Yeah.

AUDIENCE: That's a geometric series.

PROFESSOR: That's a geometric series. We just got the formula for it, so that's easy. This equals 1 minus x to the n over 1 minus x minus the 1, because I'm missing the 1 here. So we can rewrite this whole thing over here. Oops. Yikes. Got attacked. What's that?

AUDIENCE: Would it be 1 minus x should be n plus 1?

PROFESSOR: Yes it would. That's right. I've got to add 1 to there. OK. So that's good. So that says that 1 minus x times S equals 1 minus x to the n plus 1 over 1 minus x minus 1, and then I've got to remember to subtract that term too. Minus nx to the n plus 1. That means now I just divide through, and I simplify-- divide through by 1 minus x-- and simplify, and I get the following formula. I won't go through all the details, but it's not hard.

Let's see if I got that right. Yeah, that looks right. OK, so that is the closed form expression for that sum, which we can get from the perturbation method and the fact that we'd already done the geometric series.

There's another way to compute these kinds of sums, which I want to show you, because it can be useful. So we're going to do the same sum and derive the formula a different way. This method is called the derivative method, and the idea is to start with a geometric series which it's close to and then take a derivative.

So for x not equal to 1, we already know from the theorem that i equals 0 to n, x to the i equals 1 minus x to the n plus 1 over 1 minus x. That was the theorem. We already know that. Now, I can take the derivative of both sides, and let's see what we get by taking the derivative. Well, here I get the sum. The derivative of x to the i is just i times x to the i minus 1.

The derivative over here is a little messier. I've got to have-- well, I take 1 minus x times a derivative of that. Derivative of this is now minus n plus 1, x to the n. Then I take this times the derivative of that is now minus minus 1, 1 minus x to the n plus 1, and then I divide by that squared.

Now, when we compute all that out, we get this, 1 minus n plus 1, x to the n plus nx to the n plus 1 over 1 minus x squared. I won't drag you through the algebra, but it's not hard to go from there to there. This is pretty close to what we wanted. We're trying to figure out this, and we almost got there. What do I do to finish it up?

AUDIENCE: Multiply by x.

PROFESSOR: Multiply by x. Good. So if I take this and multiply by x, i equals zero to n, ix to the i equals x minus n plus 1, x to the n plus 1, plus nx to the n plus 2, all over 1 minus x squared. OK?

Which should be the same-- yeah-- the same formula we had up there. So that's called the derivative method. You can start manipulating-- you treat these things as polynomials-- these sums-- and you start manipulating them like you would polynomials. In fact, there's a whole branch of mathematics called generating functions that we won't have time to do in this class that's in chapter 12 of the text. But you do things like that to get sums. Any questions about what we did there?

You can also do a version where you take integrals of this if you want, and then you get the i's in the denominator instead of those coefficients.

For homework, I think we've given you the sum of i squared x to the i. How do you think you're going to do that? Any thoughts about how you're going to solve that? Get the sum, a closed form for the sum of i squared x to the i?

AUDIENCE: Do the derivative method twice.

PROFESSOR: Yeah, do it twice. Take this, which now you know. Take the derivative again. Won't be too hard.

You can also take the version of this where n goes to infinity. Let's do that. If the absolute value of x is less than 1, the sum i equals 1 to infinity of ix to the i, what does that equal?

This one, you can see it easier up here. What happens when n goes to infinity? What does this do? X is less than 1, an absolute value. What happens to this as n goes to infinity? This term. Goes to 0, right? This gets big, but this gets smaller faster. What happens to this term as n goes to infinity? Same thing, 0. All I'm left with is x over 1 minus x squared.

Now, this formula is useful if you're trying to, say, get the value of a company, and the company is growing. Every year the company grows its bottom line by m dollars. So the first year, the company generates m dollars, the next year it generates two m dollars in profit, the next year is three m dollars. So you've got an entity that every year is growing by a fixed amount. It's not doubling every year, but every year adds in m dollars more of profit.

What would you pay to buy that company? What is that worth? So you can think of this as, again, an annuity. Here the annuity pays im dollars, in this case, at the end, not the beginning, say, of the year i forever.

This company is-- or this annuity-- is worth, well, we just plug into the formula. Instead of $1 each year, it's m so there's an m out front. x is 1 over 1 plus p, and then we have 1 minus 1 over 1 plus p squared. And if we multiply the top and bottom by 1 plus p squared, we get m 1 plus p over p squared.

So it's possible with a very simple formula to figure out how much you should spend to buy this company, what its value is today. So, for example, say the company was adding $50,000 a year in profit. The interest rate was 6%, the value of this company is $14 million-- $14.7 million, just plugging into that formula.

So people that buy companies and stuff, they use formulas like this to figure out what it's worth. Of course, you've got to make sure it's really going to keep paying the $50,000 more every year and that this is the right interest rate to be thinking about.

You know, and the guys on Wall Street-- the bankers on Wall Street-- they all have their estimations for what these things are-- the value of p they would put into these formulas. Any questions about that? Yeah.

AUDIENCE: [INAUDIBLE]

PROFESSOR: OK. Good. So this one is OK? OK. I plugged x equals 1 over 1 plus p, like we did before-- remember for the annuity-- because every year you're degrading it, devaluing by 1 over 1 plus p. So that's the x term, and it's paying-- in the i-th year, it's paying im dollars. All right?

So the first year it pays n dollars, the next year 2 m, the next year 3 m, the next year 4 m, but every year you're knocking it down by 1 plus 1 over p to the number of years. So what you get-- the sum you've really got here-- is i equals 1 to infinity, im dollars are paid, but those dollars are worth 1 over 1 plus p to the i today, the current value. That's a good question. I should've said that. That's a great question.

So that's how we connected this up, because you're getting paid this much in our years, and that's worth that much degradation or that much devaluation today, and now we add up a total current value. So even a company that is paying you more and more every year still has a finite value, because the extra-- the payments are increasing but only linearly. The value today is decreasing geometrically, and the geometric decrease wipes out the value of the company in the future. Yeah.

AUDIENCE: Are you [? squaring ?] quantity of [INAUDIBLE].

PROFESSOR: What did I do? Oh, wait, wait, wait. I screwed up here too. Is that what you're asking about? Yeah. That's what I should have done, right? Because I got 1 minus x is 1 minus 1 over 1 plus p. That gets squared, and now when I multiply 1 plus p squared, it's multiplying this by 1 plus p. It's 1 plus p minus 1 is p. So I have p squared. All right, so this part's OK. That part I wrote wrong. That's good. Any other questions?

So let's do a simple example. What is this sum? A 1/2 plus 2/4 plus 3/8 plus 4/16 forever. What's that sum equal? You can plug that in the formula.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Yeah. That's right. Good. These details. Otherwise that sum would be what? If I didn't put the negative here, it's going to be infinity, and that's not so interesting. The negative makes it more interesting, so I got 1 over 2 to the i in there. What's it worth then? Well, I can plug in the formula. What's x? One half. So I get 1/2 over 1 minus 1/2 squared is 1/2 over 1/4, and that's 2. Any questions on that formula? It's amazing how useful these things get to be later.

So that's sort of the geometric kinds of things. Next I want to talk about more of the arithmetic kinds of sums and what you do there. In fact, we've already seen one that we've done. If I sum i equals 1 to n of i-- I think we've already done this one-- that's just n times n plus 1 over 2, and probably most of you even learned that formula back in middle school, I'm guessing-- maybe before.

How many people know the answer for this sum? The sum of the squares-- the first n squares. Somebody knows. What is it?

AUDIENCE: n times n plus 1 times 2n plus 1, all over 6.

PROFESSOR: Very good. That is correct. Most people don't remember that one. It's a little harder to derive. How would you prove this by induction? Unfortunately, induction doesn't tell you how to remember what the formula was, and there's a couple of ways you can go about that.

One is, you can remember or guess that the answer is a polynomial in n. In fact, because you're summing squares, you might guess that it's a cubic polynomial in n, and if you remember just that or guess just that, then you could actually plug in values and get the answer.

And this is-- you know, a common method of solving these sums is you sort of guess the form of the solution. In this case you might guess that for all n, the sum i equals 1 to n of i squared equals a cubic. And then what you would do is plug in the value n equals 1, n equals 2, maybe even-- we'll make it n equals 0-- make it simple and start getting some constraints on the coefficients. If you would plug in n equals 0, the sum is 0.

The polynomial evaluates to d. That tells you what d has got to be right away. n equals 1. The sum is 1, and when you plug into the polynomial, you get a plus b plus c plus d. n equals 2. Well, that's 1 plus 4 is 5, 2 cubed is 8, so you have 8a plus 4b plus 2c plus d, and you'll need one more since you've got four variables. Let's see, 1 plus 4 plus 9 is 14. I've now got 3 cubed is 27a plus 9b plus 3c plus d.

So now I've got four equations and four variables and, with any luck, I can solve that system of equations and get the answer. And, in fact, you can. When you solve this system, you get a equals 1/3, b equals 1/2, c equals 1/6, and d equals 0. And that's exactly what you get in that formula. So that's a way to reproduce the formula if you forgot it.

Now, this method-- really to be sure you got the right answer-- you've got to go prove it by induction, because I derived the answer-- if it was a polynomial, I would have gotten it right, but I might be wrong in my guess. And to make sure your guess is right, you've got to go back and use induction to prove it for this approach. Yeah.

AUDIENCE: How do you know that it would be [INAUDIBLE] and not some higher power?

PROFESSOR: Well, it turns out that anytime you're summing powers, the answer is a polynomial to one higher degree. So if you just remembered that fact, or you guessed that fact. Another way to sort of imagine that might be true is that I'm getting n of them, so I might be multiplying-- the top one is n squared, so I'm going to have n of them about n squared. Might be something like n cubed. That's another way you could think of it, to guess that. Any other questions on this?

So far, all these sums had nice closed forms, and a lot of them do that you'll encounter later on, but not all, and sometimes you get sums that don't have a nice closed form-- at least nobody has ever figured out one, and probably doesn't always exist. For example, what if I want to sum the first n square roots of integers? Let's write that down.

So say I want a closed form for this guy. Nobody knows an answer for that, but there are ways of getting very good, close bounds on it that are closed form, and these are very important, and we're going to use this the rest of today and the rest of next time. And they're based on replacing the sum with an integral, and the integral is very close to the right answer, and then we can see what the error terms are.

So let's first look at the case when we've got a sum where the terms are increasing as i grows, and we'll call these integration bounds, and a general sum will look like this-- i equals 1 to n of f of i, and the first case is when f is a positive increasing function, increasing in i. Integration bounds, and so we're increasing function. So let me draw a picture that will hopefully make the bounds that we're going to get pretty easy.

So let's draw the sum here as follows. I've got 0, 1, 2, 3, n minus 2, n minus 1, n, and draw the values of f here. Here's f of 1, f of 2-- it's increasing-- f of 3, f of n minus 1, and f of n. Then I'll draw the rectangles here. So this has area of f of 1, this has area f of 2, this has area f of 3, and we keep on going. Let's see, this will be f of n minus 2 on this one-- I'll just do f of n minus 1, draw this guy here.

So its unit width, its height is f of n minus 1, so its area is f of n minus 1, and then f of n. And let me also-- so the sum of f of i is the areas in the rectangles. That's what the sum is, and I want to get bounds on this sum using the integral, because integrals are easier to compute.

So let's draw the function f of x from 1 to n. All right, so this is f of x as a function. Now I claim that the sum i equals 1 to n of f of i is at least f of 1 plus the integral from 1 to n, f of x, dx.

Now, the integral from 1 to n of f of x is this stuff, the stuff under the curve. It comes down here, starts at 1, and it's the stuff under the curve. And what I'm saying here is that if you take that stuff under the curve and add f of 1, which is this piece, that's a lower bound on our sum. The sum's bigger than that. So what I'm saying is the area in the rectangles is at least as big as the area in the first rectangle plus the area under the curve. Does everybody see why that is?

I'm saying the sum is the area in the rectangles, right? That's pretty clear. And that is at least as big as the first rectangle f of 1 plus the stuff under the curve, which is the integral, and I've left-- I've chopped off these guys. That's extra. OK? Is that all right? So lower bound. Any questions on the lower bound?

This is a picture proof, which we always tell you not to do, but we're going to do one here. And, of course, it totally hides why did I need f is increasing, but we'll see that in a minute. The proof would not work unless it is increasing here. Any questions, because now going I'm going to do the other bound, the other side.

I also claim-- this will be a little trickier to see-- that the sum is at most f of n plus the integral from 1 to n. So this is the lower bound add in f of 1, the upper bound just add in f of n. So let's see why that's true. Now, to see that, this is-- I'm not going to be able to draw it. I want you to imagine taking this curve and the area under it, down to here, and sliding it left one unit.

sliding it left to here, sliding it left one unit over to here. Now, when I slide it left one unit, did the area under it change? No. It's the same area under it, just where it sits on the picture is now out here. It's this area under this guy, but it's the same thing, its the same integral. And you can see that it's more than what's in these rectangles, because I got all this stuff.

And, of course, I didn't even include this, so now I add the f n. So if I take the area under the curve, which is the integral, shift it left one, so it only goes up to here now, and then add in this rectangle, that dominates the area in the rectangles. Bigger than. Do you see that? I could do a lot of equations on the board but, for sure, that would be hopeless to follow. Any questions about this? Yeah.

AUDIENCE: I guess I understand the lower amounts because we're cutting off the triangles.

PROFESSOR: Yeah.

But is there-- is it a lot of hand waving, or am I just missing something, that it's always going to be the f of n is what we--

PROFESSOR: Yeah. There's a little hand waving going on, but I do believe it is true. With equations you can make it precise. Let's look at it again, do this one more time.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Yeah, I'm waving hands a little bit, but it also-- hopefully the intuition comes across, because when I shift left by one, this point becomes this point, and that point becomes that point, and I'm catching sort of the cusp of these rectangles, and now I take this curve here, shifting the whole curve left one unit, and the area doesn't change. It would be equivalent of taking the integral from 0 to n minus 1. You notice what I'm doing-- of f of x plus 1. There's another way to look at it mathematically, maybe-- probably the formal way to do it-- but the idea is that this now contains all these first n minus 1 rectangles, are contained underneath it. And then I just add this in, and I'm good to go. I've contained all the rectangles now for the upper bounds. All right, mathematically what this is, this curve is 0 to n minus 1, fx plus 1, which is the same thing as what that is. Any questions for that?

But now I have good bounds on the sum. We know that the sum is at least this and at most that, and those two bounds only differ by a single term in the sum, so they're very close. These actually are good formulas to write down for your crib sheet. That is worth doing on the test, and we'll use them more today, and we'll also use them on Thursday.

So now we can actually get close bounds on the sum of the square roots. So let's see how this works for f of i equal to the square root of i, which is increasing. So I compute the integral from 1 to n of square root of x dx, and that's just x to the 3/2 over 3/2, evaluated at n and 1, and that just equals 2/3 n to the 3/2 minus 1.

And now I can compute the bounds on the sum of the first n square roots. So I know that square root i equals 1 to n-- well, the upper bound is f n plus the integral, and the lower bound is f of 1 plus the integral. What is f of n? That's the square root of n. What is f of 1? The square root of 1, which is 1. So I'll plug that in here. I get 2/3 n to the 3/2 plus 1 minus 2/3 is plus 1/3, and here I get 2/3 n to the 3/2 plus the square root of n minus 2/3. So now I have pretty good bounds on the sum of the first n square roots using this method.

So for example, take n equals 100. This number evaluates to 667. This number evaluates to 676, and the difference is 9, which is the square root of 100 minus 1. This is the square root of n, this is 1, so the gap here, square root of n minus 1, and n equals 100. Square root of 100 minus 1 is 9, so I shouldn't be surprised my gap here is nine.

So I didn't get exactly the right answer, but I'm pretty close here. Now, as n grows, what happens to the gap between the upper bound and the lower bound? What does it do? It gets bigger. So my gap gets bigger. That's not so nice. Doesn't always stay 9 forever, but somehow though this is still pretty good, because the gap grows but the gap only grows as square root of n, where the answer-- the bounds-- are growing as n to the 3/2.

In other words, my error is somewhere around here, and that gets smaller compared to my answer, which is somewhere around there. And there's a special notation that people use. In fact, let's write that down and then do the notation.

Another way of writing this is that the sum i equals 1 to n of square root of i. The leading term here is 2/3 n to the 3/2, and then there's some error term-- delta term here. We'll call that delta n, and we know that the error term is at least 1/3 and, at most, the square root of n minus 2/3. So this delta term is bound by the square root of n. That's getting bigger as n gets big, but this value compared to your answer is getting small. That's nice, and so the way that gets represented is as follows.

We would say-- it's using that tilde notation. We write tilde 2/3 times n to the 3/2, and now we've gotten rid of the delta, because this tilde is telling us that everything else out here gets small compared to this as n gets big. And the formal definition-- Let's write out the formal definition for it. Now, a lot of times you'll see people use this symbol to mean about. That's informal. When I'm using it here, it's a very formal meaning mathematically. A function g of x is tilde, a function h of x means that the limit as x goes to infinity of g over h is 1.

In other words, that as x goes to infinity-- as x gets big-- the ratio of these guys becomes 1. And let's see if that's true here. Well, square root of i equals this, so I need to show the limit of this over that is 1. So let's check that. A limit as n goes to infinity of 2/3 n to the 3/2 plus that delta term over 2/3 n to the 3/2. Well, that equals-- divide out by 2/3 n to the 3/2, I get a 1.

Did I-- should I have subtracted? No, that's OK. One plus delta n over 2/3 n to the 3/2. If I can pull the 1 out front, that's 1 plus the limit, and this is now delta n is at most square root of n, so I get square root of n over 2/3 n to the 3/2. Square root of n over n to the 3/2, that goes to 0. So this equals 1. So this limit is 1, and so therefore I can say that the sum of the first n square roots is tilde 2/3 n to the 3/2. Any questions about this? We're going to do a lot of this kind of notation next time. Yeah.

AUDIENCE: When you took the integral and got 2/3 into the 3/2 minus 1, why did the minus 1 not become part of the actual solution and become part of the delta?

PROFESSOR: OK. So you brought it up to here, right? OK. So then I plug that into the integral that appears on both sides here, and here I add the f 1, here I add f n, and now I have the lower bound here and the upper bound here. Are you good with those? All right.

Now some judgment takes place, and what I'm really trying to do here is figure out what are the important terms in these bounds as n gets big? How big is this growing as n gets big? Well, as n gets big, the 1/3 is not doing much. As n gets big, the square root of n grows, but it's nothing like what's happening here.

If you had to describe to somebody what's going on in this bound, would you start here or here? No. You'd start here. This is the action, and there's a little bit-- the rest is just in the slop, in the air. And so now I've used judgement to say that delta is somewhere in this stuff here.

What's really happening, and the nice thing is they match. The lower bound and the upper bound match on that term. So what I do is I write it equals this term plus something that's smaller and, in particular, it's between 1/3 and square root of n minus 2/3. All right.

So what I'm trying to capture here is just the guts of what's happening to this function as n grows, and the guts of it is this. It's not exactly equal to 2/3 times n to the 3/2, but it's close and, in fact, if I take the limit of this over that, that limit goes to 1.

It's a way of saying they're approximately the same that's called asymptotically the same, and we'll talk a lot more about asymptotic notation next time. We'll give you five more symbols besides tilde that people use. Any questions about-- maybe start with the bounds. Any question on the bounds that we got? That's the integration method, first getting the bounds. You take the integral, you add f of 1 for the lower bound, you add f of n for the upper bound, and it's in between, somewhere in there. Questions there?

All right. Then we plugged it in, and now we look at this tilde notation that says-- well, first we'd write it like this. The sum is this value plus an error term and, lo and behold, that error term is small. If I take the limit of the whole thing divided by the big term, I get 1, which means this thing is really not important. So I write this. Questions on that? All right.

There's one more case to consider, and now we're going to go back to the integration bounds, and that is when f is a decreasing function, and we're going to do the analysis for that, and then we'll be all done.

So we're going to look at integration bounds when f is decreasing and positive still. The example here might be, for example, the sum i equals 1 to n, 1 over the square root of i. Say you had to get some idea of how fast that function is growing as a function of n. I'm summing the first n inverse as the square roots. What is that roughly going to be? How fast does that grow as a function of n? So let's do that.

And, of course, 1 over square root of i decreases as i gets bigger. So let's do the general picture again and see what happens. So we have 0, 1, 2, 3, n minus 2, n minus 1, n, and now f of n is the small term and f of n minus 1, f of 3 here, f of 1.

Now, I'm going to draw the rectangle. This has area f of 1. This one has area f of 2. This one has area f of 3, and then this one has area f of n minus 1 here, and then, lastly, area f of n. So the sum is the area in the rectangles, just like before, except now the rectangles are getting smaller. Let's draw the integral like we did before. The integral is the area under this curve, f of x here, just like before. So this is f of x, only it's decreasing.

Now, let's take the area under this curve, and add f of 1 to it. If I take the area under this curve, all the way down to here and then add f of 1, what do I get? Upper bound on my sum. OK. So the sum i equals 1 to n f of i is upper bounded by that guy, which is f of 1, plus my integral, which is the area under the curve. The integral is the area under that curve, starting here, and that contains all these rectangles, and then I just add in f of 1 to get an upper bound.

Now, for the lower bound, think about shifting the whole curve left by one. That goes to there, this goes to here, that goes to here, that goes to there, and that goes to there. The area under the curve did not change when I shifted it left by one. This is now my area under the curve. Stops here. What do I get when I take that area and add in this last box, f of n? A lower bound, because it's contained in all the rectangles.

Now, what's really weird about these formulas, do they look familiar? Yeah. Yeah. I switched them. Yeah. They're the same formulas we had over here, except we switched the direction on the less than and greater than signs. Well, I swapped f of 1 and f of n, however you want to think about it. The lower bound here in that case became the upper bound in this case. Is that possible that the lower bound became the upper bound? Yeah. Yeah, because what really happened here-- which is the big term in this case, fn or f1? f1 is the big term because it's decreasing, so it's totally symmetric. All right?

The proof was very similar, so the nice thing is you've only got to remember the bounds are now simple for any sum as long as an increasing or decreasing, it's the same as the integral. The lower bound is the smaller of the first and last term, and the upper bounds are larger of the first and last term. Very easy to remember. Probably don't even need the crib sheet for it, although to be safe, want to write that down. So now it's easy to compute good bounds on the sum of the inverse square roots. Any questions there before I go do it?

So let's take the case where we're summing 1 over square root of i. So we compute the integral of 1 over square root of x dx. That equals the square root of x over 1/2, evaluated at n and 1. That equals 2 square root of n minus 1, or two square root of n minus 2, and now we can bound the sum. The upper bound is f of 1 plus the integral. The lower bound is f of n plus the integral. What is f of 1? One? One over the square root of 1 is 1. What is f of n? One over the square root of n. Small.

So these bounds are pretty close here, right? In fact, this gets really tiny as n gets big, so I'm just going to replace this with 2 square root of n minus 2 and make it a strict lower bound, and this-- cancel there-- I get 2 square root of n minus 1. Wow, these bounds are great. They're within one for all n. That's really good.

So we can rewrite this in terms of what really matters. What really matters in these bounds? How fast is this function growing?

AUDIENCE: 2 root n.

PROFESSOR: 2 root n. That's what really matters, so let's write that down. So this says that the sum i equals 1 to n of 1 over square root i equals 2 square root of n-- we have a minus delta n, where delta is between 1 and 2. And so if I use the tilde notation, what would I write down here for the tilde? Past the tilde? I don't want to mess--

AUDIENCE: Tilde.

PROFESSOR: --I don't want to keep track of all the delta stuff as n gets big.

AUDIENCE: 2 root n.

PROFESSOR: 2 root n, because this term over that goes to 0 as n gets large, so let's just check that. So we take the limit as n goes to infinity of 2 root n minus delta n over 2 root n. I'm just checking the definition now. That's what the definition would be. Equals 1 minus the limit as n goes to infinity of 2 over 2 root n. This is 0. So it equals 1. And so now you know that the sum of the first n inverse square roots grows as 2 root n, which is the integral. Yeah.

AUDIENCE: [INAUDIBLE] dropped off the lower bound that f of n was 1 over root n?

PROFESSOR: Yeah, I dropped it off, because it was so tiny and going to zero, I just made a strict less than.

In fact, yes. I don't hurt myself by dropping it off. In fact, the lower bound was a little bit-- I made a little weaker lower bound. So this is still true. I just-- it wasn't as tight as it used to be, so I could keep it around.

Yeah, it doesn't hurt to keep it around, then it's a less than or equal there. And now this would be something like that. So I could keep it around, but I'm going to get rid of it anyway, because I'm going to go to the tilde notation, and as n gets big, this is really tiny. So in this case, the bounds are great. You can nail it pretty much right on. Yeah.

AUDIENCE: You said one over n still there, the number is bigger than it would normally be, so when you take it out, it becomes smaller, so how could you go to a less than [INAUDIBLE]?

PROFESSOR: Well, you're saying you don't like the fact I dropped it here?

AUDIENCE: Yes. When you drop it, why do you go to a less than instead of [INAUDIBLE]?

PROFESSOR: Oh, because I've got a bigger bound that I made less when I dropped it. I took something away, so I know I could never equal this, because I know it's bigger than this. I know that the real answer has to be at least this big, and so it has to be bigger than something smaller. That's why I did it. Any other questions? We'll get more practice tomorrow and next time with this stuff.