Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science
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6.050J/2.110J **Information and Entropy** **Spring 2003**

Problem Set 11 Solutions

Solution to Problem 1: Cheap Heat

Solution to Problem 1, part a.

You must run it in the reverse direction.

Solution to Problem 1, part b.

The outdoor temperature in Kelvin is 273.15 degrees, and the indoor temperature is 290 degrees Kelvin.

Solution to Problem 1, part c.

To find a relationship between T_1, T_2, H_c, and H_d we take the equation given and

$$T dS = \left(\sum_i p_i (E_i(H) - E)^2 \right) \frac{1}{k_B T} \left(\frac{1}{T} dT - \frac{1}{H} dH \right) \tag{11–4}$$

but since $dS = 0$ the equation reduces to

$$\frac{1}{T} dT = \frac{1}{H} dH$$

integrating from c to d we have

$$\int_{T_2}^{T_1} \frac{1}{T} dT = \int_{H_c}^{H_d} \frac{1}{H} dH$$

$$\ln \left(\frac{T_1}{T_2} \right) = \ln \left(\frac{H_d}{H_c} \right)$$

$$\frac{T_1}{T_2} = \frac{H_d}{H_c} \tag{11–5}$$

Solution to Problem 1, part d.

Thus finding H_d we have

$$\begin{aligned} H_d &= H_c \frac{T_1}{T_2} \\ &= 1000 \times \frac{273.15}{290} \\ &= 942 \text{ A/m} \end{aligned} \tag{11–6}$$

(11–7)

Solution to Problem 1, part e.

The heat extracted from outdoors is

$$Q = (S_2 - S_1)T_1 \tag{11–8}$$

The work done on the system is the heat pumped to the warm environment less the heat extracted from the cold environment

$$W = \frac{S_2 - S_1}{T_2 - T_1} \tag{11–9}$$

The coefficient of performance then is

$$\begin{aligned} \eta &= \frac{T_1}{T_2 - T_1} \\ &= \frac{273.15}{290 - 273.15} \\ &= 16.21 \end{aligned} \tag{11–10}$$

Solution to Problem 1, part f.

Again, to find a relationship between T_1, T_2, H_a, and H_b we take the equation given

$$TdS = \left(\sum_i p_i(E_i(H) - E)^2\right)\frac{1}{k_B T}\left(\frac{1}{T}dT - \frac{1}{H}dH\right) \tag{11–11}$$

but since $dS = 0$ the equation reduces to

$$\frac{1}{T}dT = \frac{1}{H}dH$$

integrating from a to b we have

$$\begin{aligned} \int_{T_1}^{T_2} \frac{1}{T}dT &= \int_{H_a}^{H_b} \frac{1}{H}dH \\ \ln\left(\frac{T_2}{T_1}\right) &= \ln\left(\frac{H_b}{H_a}\right) \\ \frac{T_2}{T_1} &= \frac{H_b}{H_a} \end{aligned} \tag{11–12}$$

Solution to Problem 1, part g.

The magnetic field H_a is

$$\begin{aligned} H_a &= H_b\frac{T_1}{T_2} \\ &= 3000 \times \frac{273.15}{290} \\ &= 2826 \text{ A/m} \end{aligned} \tag{11–13}$$

$$\tag{11–14}$$

Solution to Problem 1, part h.

Since S is constant in this (adiabatic) leg, $dq = 0$.

To go further you have to calculate the probabilities, since you need them to find the energy E at each of the four corners. You already know the temperature and magnetic field at each corner, so it is straightforward to find α and then the probabilities using these equations from Chapter 13:

$$p_i = e^{-\alpha} e^{-E_i/k_B T} \tag{11–15}$$

$$\alpha = \ln\left(\sum_i e^{-E_i/k_B T}\right) \tag{11–16}$$

Because the magnetic energy is so small compared with thermal energy $k_B T$, the probabilities are all very close to 0.5. You may find it necessary to retain a lot of significant figures, or else use a suitable approximation.

Solution to Problem 1, part i.

For corners a and b:

$$p_{up} = \frac{e^{m_d H_a/k_B T_1}}{e^{m_d H_a/k_B T_1} + e^{-m_d H_a/k_B T_1}} \tag{11–17}$$

First calculate the exponential.

$$\begin{aligned} e^{m_d H_a/k_B T_1} &= \exp\left(\frac{1.165 \times 10^{-29} \times 2826}{273.15 \times 1.38 \times 10^{-23}}\right) \\ &= \exp\left(\frac{3.29 \times 10^{-26}}{3.77 \times 10^{-21}}\right) \\ &= \exp(8.726 \times 10^{-6}) \\ &= 1 + 8.7328 \times 10^{-6} \end{aligned} \tag{11–18}$$

Thus. . .

$$\begin{aligned} p_{up,a,b} &= \frac{1 + 8.7328 \times 10^{-6}}{1 + 8.7328 \times 10^{-6} + \frac{1}{1+8.7328\times10^{-6}}} \\ &= \frac{1 + 8.7328 \times 10^{-6}}{1 + 8.7328 \times 10^{-6} + 1 - 8.7328 \times 10^{-6}} \\ &= 0.5 + 4.3664 \times 10^{-6} \end{aligned} \tag{11–19}$$

$$\begin{aligned} p_{down,a,b} &= 1 - p_{up} \\ &= 0.5 - 4.3664 \times 10^{-6} \end{aligned} \tag{11–20}$$

For corners c and d:

$$p_{up} = \frac{e^{m_d H_d/k_B T_1}}{e^{m_d H_d/k_B T_1} + e^{-m_d H_d/k_B T_1}} \tag{11–21}$$

First calculate the exponential. . .

$$\begin{aligned} e^{m_d H_a/k_B T_1} &= \exp\left(\frac{1.165\times10^{-29}\times942}{273.15\times1.38\times10^{-23}}\right) \\ &= \exp\left(\frac{1.097\times10^{-26}}{3.77\times10^{-21}}\right) \\ &= \exp(2.91\times10^{-6}) \\ &= 1+2.9109\times10^{-6} \end{aligned} \tag{11–22}$$

Thus...

$$\begin{aligned} p_{up,c,d} &= \frac{1+2.9109\times10^{-6}}{1+2.9109\times10^{-6}+\frac{1}{1+2.9109\times10^{-6}}} \\ &= \frac{1+2.9109\times10^{-6}}{1+2.9109\times10^{-6}+1-2.9109\times10^{-6}} \\ &= 0.5+1.455\times10^{-6} \end{aligned} \tag{11–23}$$

$$\begin{aligned} p_{down,c,d} &= 1-p_{up} \\ &= 0.5-1.455\times10^{-6} \end{aligned} \tag{11–24}$$

Solution to Problem 1, part j.

$$
\begin{aligned}
E_a &= \sum_i E_i p_i \\
&= -m_d H_a p_{up,a} + m_d H_a p_{down,a} \\
&= m_d H_a (p_{down,a} - p_{up,a}) \\
&= -1 \times 1.165 \times 10^{-29} \times 2825(4.3644 + 4.3644 \times 10^{-6}) \\
&= -2.87 \times 10^{-31} \text{ Joules} && (11\text{–}25) \\
E_b &= \sum_i E_i p_i \\
&= -m_d H_b p_{up,b} + m_d H_b p_{down,b} \\
&= m_d H_b (p_{down,b} - p_{up,b}) \\
&= -1 \times 1.165 \times 10^{-29} \times 3000(4.3644 + 4.3644 \times 10^{-6}) \\
&= -3.05 \times 10^{-31} \text{ Joules} && (11\text{–}26) \\
E_c &= \sum_i E_i p_i \\
&= -m_d H_c p_{up,c} + m_d H_c p_{down,c} \\
&= m_d H_c (p_{down,c} - p_{up,c}) \\
&= -1 \times 1.165 \times 10^{-29} \times 1000(2 \times 1.455 \times 10^{-6}) \\
&= -3.39 \times 10^{-32} \text{ Joules} && (11\text{–}27) \\
E_d &= \sum_i E_i p_i \\
&= -m_d H_d p_{up,d} + m_d H_d p_{down,d} \\
&= m_d H_d (p_{down,d} - p_{up,d}) \\
&= -1 \times 1.165 \times 10^{-29} \times 942(2 \times 1.455 \times 10^{-6}) \\
&= -3.19 \times 10^{-32} \text{ Joules} && (11\text{–}28) \\
& && (11\text{–}29)
\end{aligned}
$$

Solution to Problem 1, part k.

$$
\begin{aligned}
S_1 &= k_B \sum_i p_i \ln\left(\frac{1}{p_i}\right) \\
&= k_B \left(p_{up,a,b} \ln\left(\frac{1}{p_{up,a,b}}\right) + p_{down,a,b} \ln\left(\frac{1}{p_{down,a,b}}\right)\right) \\
&= k_B \left((0.5 + 4.3664 \times 10^{-6})(0.69313845179804106245058732398961) + (0.5 - 4.3664 \times 10^{-6})(0.69315590939804150\right. \\
&= k_B(0.34657225241875646219178874623899 + 0.34657492810305795730495872070288) \\
&= 0.69314718052181441949674746694099 k_B \qquad (11\text{–}30) \\
S_2 &= k_B \sum_i p_i \ln\left(\frac{1}{p_i}\right) \\
&= k_B \left(p_{up,c,d} \ln\left(\frac{1}{p_{up,c,d}}\right) + p_{down,c,d} \ln\left(\frac{1}{p_{down,c,d}}\right)\right) \\
&= k_B \left((0.5 + 1.455 \times 10^{-6})(0.69314427056417935120319304859584) + (0.5 - 1.455 \times 10^{-6})(0.69315009056417936763\right. \\
&= k_B(0.34657314380700334648255252494339 + 0.34657403674870791293467362078774) \\
&= 0.69314718055571125941722614573074 k_B \qquad (11\text{–}31)
\end{aligned}
$$

therefore

$$
\begin{aligned}
S_2 - S_1 &= (0.69314718055571125941722614573074 - 0.69314718052181441949674746694099) k_B \\
&= 3.39 \times 10^{-11} k_B
\end{aligned}
\qquad (11\text{–}32)
$$

Solution to Problem 1, part l.

$$
\begin{aligned}
dq_{ba} &= T dS \\
&= 0 \text{ Joules} \qquad (11\text{–}33) \\
dq_{ad} &= T dS \\
&= T_2(S_1 - S_2) \\
&= 290 \times 3.39 \times 10^{-11} k_B \qquad (11\text{–}34) \\
&= -1.356 \times 10^{-31} \text{ Joules} \qquad (11\text{–}35) \\
dq_{dc} &= T dS \\
&= 0 \text{ Joules} \qquad (11\text{–}36) \\
dq_{cb} &= T dS \\
&= T_1(S_2 - S_1) \\
&= 273.15 \times -3.39 \times 10^{-11} k_B \qquad (11\text{–}37) \\
&= 1.277 \times 10^{-31} \text{ Joules} \qquad (11\text{–}38) \\
& \qquad (11\text{–}39)
\end{aligned}
$$

Solution to Problem 1, part m.

$$\begin{aligned}
dw_{ba} &= dE_{ba} - dq_{ba} \\
&= E_b - E_a - 0 \\
&= -3.05 \times 10^{-31} + 2.87 \times 10^{-31} \\
&= 1.77 \times 10^{-31} \text{ Joules} \\
dw_{ad} &= dE_{ad} - dq_{ad} \\
&= E_a - E_d - dq_{ad} \\
&= -2.87 \times 10^{-31} + 3.19 \times 10^{-32} + 1.356 \times 10^{-31} \\
&= -1.35 \times 10^{-31} \text{ Joules} \\
dw_{dc} &= dE_{dc} - dq_{dc} \\
&= E_d - E_c - 0 \\
&= -3.19 \times 10^{-32} + 3.39 \times 10^{-32} \\
&= -2.0 \times 10^{-33} \text{ Joules} \\
dw_{cb} &= dE_{cb} - dq_{cb} \\
&= E_c - E_b - dq_{cb} \\
&= -3.39 \times 10^{-32} + 3.05 \times 10^{-31} - 1.277 \times 10^{-31} \\
&= 1.434 \times 10^{-31} \text{ Joules}
\end{aligned} \tag{11–40}$$

Solution to Problem 1, part n.

The work is the sum of the previous.

$$1.77 \times 10^{-31} - 1.35 \times 10^{-31} - 2.0 \times 10^{-33} + 1.434 \times 10^{-31} = 1.834 \times 10^{-31} \tag{11–41}$$

Solution to Problem 1, part o.

$$\frac{1.277 \times 10^{-31}}{1.834 \times 10^{-31}} = 0.69 \tag{11–42}$$

This is a very interesting number, which does not compare favorably with the coefficient of performance.

Solution to Problem 1, part p.

The number of Joules required to heat one gram of air one degree is

$$\frac{0.715}{1.277 \times 10^{-31}} = 5.59 \times 10^{30} \text{ cycles} \tag{11–43}$$

Solution to Problem 1, part q.

$$\frac{5.59 \times 10^{30}}{6.023 \times 10^{23}} = 9.29 \times 10^{6} \text{ cycles} \tag{11–44}$$

Solution to Problem 2: Information is Cool

Solution to Problem 2, part a.

$$\frac{75\ \text{Calories/hour} \times 4.1868 \times 10^3\ \text{Joules/Calorie}}{3600\ \text{sec/hour}} = 87.225\ \text{Joules/sec} \tag{11–45}$$

People don't light up like lightbulbs because the energy they expend is distributed about the whole body, not concentrated on a microscopic filament.

Solution to Problem 2, part b.

$$75\ \text{Calories/day} \times 24\ \text{hours/day} = 1800\ \text{Calories/day} \tag{11–46}$$

Solution to Problem 2, part c.

The jogger will expend

$$75\ \text{Calories/hour} \times 23.5\ \text{hours/day} + 550\ \text{Calories/hour} \times 0.5\ \text{hours} = 2037.5\ \text{Calories/day} \tag{11–47}$$

The difference is 2037.5 - 1800 = 237.5 Calories/day, which over thirty days accumulates to 7125 Calories. If all of this is stored as fat, we get

$$\frac{7125 Calories \times 4.1868 \times 10^3\ \text{Joules/Calorie}}{33.1 \times 10^6\ \text{Joules/kg fat}} = 0.9012\ \text{kg fat} \tag{11–48}$$

Solution to Problem 2, part d.

The amount of heat the room is losing, in Watts, is:

$$\frac{9000 \times 10^3 \text{Joules/hour}}{3600\ \text{sec/hour}} = 2500\ \text{Watts} \tag{11–49}$$

If the temperature is to remain constant, the students and professor must produce the same amount of energy

$$200 + 100A + 70S = 2500\ \text{Watts} \tag{11–50}$$

where A = Awake and S = Sleeping. If the lecture has 24 students, then the sum of A and S equals 24 and so

$$\begin{aligned} 1880 + 30A &= 2500 \\ A &= 20.6\ \text{students} \end{aligned} \tag{11–51}$$

which means that 20 students must be awake, 3 students asleep, and one student drifting in and out of consciousness, his head bobbing forward, waking himself up every so often, for an average of 66% of the time awake, 33% asleep.

Solution to Problem 2, part e.

The number of Calories consumed in raising 335 ml of water to body temperature (37 degrees Celsius) is

$$0.355\ \text{Liter} \times 1\ \text{Calories/Liter/degree C} \times 37 \text{degrees C} = 13\ \text{Calories} \tag{11–52}$$

Only 7% of the Calories are consumed raising the rootbeer to body temperature. So Paul's argument is not correct.