6.231 DYNAMIC PROGRAMMING

LECTURE 20

LECTURE OUTLINE

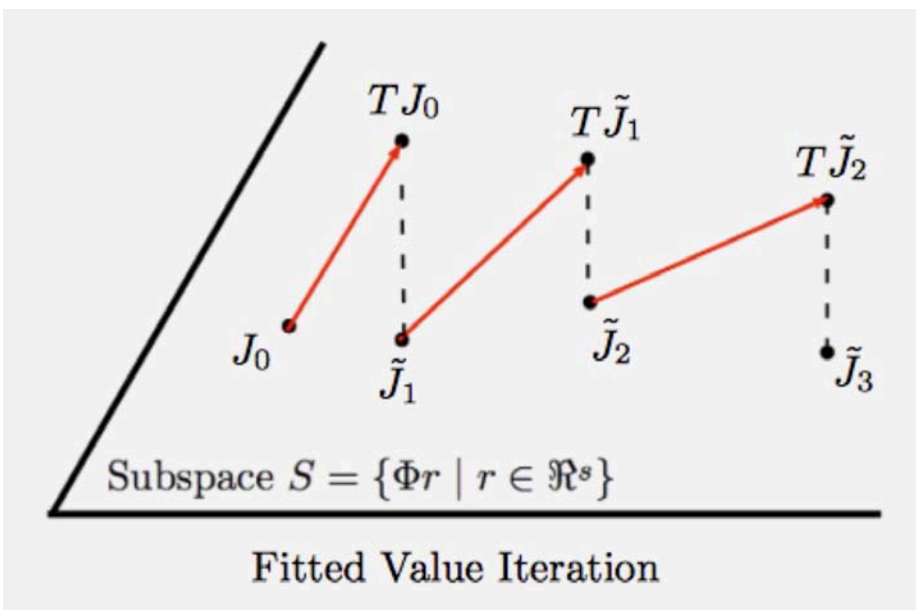
- Discounted problems - Approximation on subspace $\{\Phi r \mid r \in \Re^s\}$
- Approximate (fitted) VI
- Approximate PI
- The projected equation
- Contraction properties - Error bounds
- Matrix form of the projected equation
- Simulation-based implementation
- LSTD and LSPE methods

REVIEW: APPROXIMATION IN VALUE SPACE

- Finite-spaces discounted problems: Defined by mappings T_μ and T ($TJ = \min_\mu T_\mu J$).
- Exact methods:
  - VI: $J_{k+1} = TJ_k$
  - PI: $J_{\mu^k} = T_{\mu^k} J_{\mu^k}, \quad T_{\mu^{k+1}} J_{\mu^k} = TJ_{\mu^k}$
  - LP: $\min_J c'J$ subject to $J \leq TJ$
- Approximate versions: Plug-in subspace approximation with Φr in place of J
  - VI: $\Phi r_{k+1} \approx T\Phi r_k$
  - PI: $\Phi r_k \approx T_{\mu^k} \Phi r_k, \quad T_{\mu^{k+1}} \Phi r_k = T\Phi r_k$
  - LP: $\min_r c'\Phi r$ subject to $\Phi r \leq T\Phi r$
- Approx. onto subspace $S = \{\Phi r \mid r \in \Re^s\}$ is often done by projection with respect to some (weighted) Euclidean norm.
- Another possibility is aggregation. Here:
  - The rows of Φ are probability distributions
  - $\Phi r \approx J_\mu$ or $\Phi r \approx J^*$, with r the solution of an "aggregate Bellman equation" $r = DT_\mu(\Phi r)$ or $r = DT(\Phi r)$, where the rows of D are probability distributions

APPROXIMATE (FITTED) VI

- Approximates sequentially $J_k(i) = (T^k J_0)(i)$, $k = 1, 2, \ldots$, with $\tilde{J}_k(i; r_k)$
- The starting function J_0 is given (e.g., $J_0 \equiv 0$)
- Approximate (Fitted) Value Iteration: A sequential "fit" to produce $\tilde{J}_{k+1}$ from $\tilde{J}_k$, i.e., $\tilde{J}_{k+1} \approx T\tilde{J}_k$ or (for a single policy μ) $\tilde{J}_{k+1} \approx T_\mu \tilde{J}_k$



- After a large enough number N of steps, $\tilde{J}_N(i; r_N)$ is used as approximation to $J^*(i)$
- Possibly use (approximate) projection Π with respect to some projection norm,

$$\tilde{J}_{k+1} \approx \Pi T \tilde{J}_k$$

WEIGHTED EUCLIDEAN PROJECTIONS

- Consider a weighted Euclidean norm

$$\|J\|_\xi = \sqrt{\sum_{i=1}^{n} \xi_i \big(J(i)\big)^2},$$

where $\xi = (\xi_1, \ldots, \xi_n)$ is a positive distribution ($\xi_i > 0$ for all i).

- Let Π denote the projection operation onto

$$S = \{\Phi r \mid r \in \Re^s\}$$

with respect to this norm, i.e., for any $J \in \Re^n$,

$$\Pi J = \Phi r^*$$

where

$$r^* = \arg\min_{r \in \Re^s} \|\Phi r - J\|_\xi^2$$

- Recall that weighted Euclidean projection can be implemented by simulation and least squares, i.e., sampling $J(i)$ according to ξ and solving

$$\min_{r \in \Re^s} \sum_{t=1}^{k} \big(\phi(i_t)' r - J(i_t)\big)^2$$

FITTED VI - NAIVE IMPLEMENTATION

- Select/sample a "small" subset I_k of representative states
- For each $i \in I_k$, given $\tilde{J}_k$, compute

$$(T\tilde{J}_k)(i) = \min_{u \in U(i)} \sum_{j=1}^{n} p_{ij}(u)\big(g(i,u,j) + \alpha \tilde{J}_k(j;r)\big)$$

- "Fit" the function $\tilde{J}_{k+1}(i;r_{k+1})$ to the "small" set of values $(T\tilde{J}_k)(i)$, $i \in I_k$ (for example use some form of approximate projection)
- "Model-free" implementation by simulation
- Error Bound: If the fit is uniformly accurate within $\delta > 0$, i.e.,

$$\max_i |\tilde{J}_{k+1}(i) - T\tilde{J}_k(i)| \leq \delta,$$

then

$$\lim \sup_{k \to \infty} \max_{i=1,\ldots,n} \big(\tilde{J}_k(i, r_k) - J^*(i)\big) \leq \frac{\delta}{1-\alpha}$$

- But there is a potential serious problem!

AN EXAMPLE OF FAILURE

- Consider two-state discounted MDP with states 1 and 2, and a single policy.
  - Deterministic transitions: $1 \to 2$ and $2 \to 2$
  - Transition costs $\equiv 0$, so $J^*(1) = J^*(2) = 0$.

- Consider (exact) fitted VI scheme that approximates cost functions within $S = \{(r, 2r) \mid r \in \Re\}$ with a weighted least squares fit; here $\Phi = (\,1, 2\,)'$

- Given $\tilde{J}_k = (r_k, 2r_k)$, we find $\tilde{J}_{k+1} = (r_{k+1}, 2r_{k+1})$, where $\tilde{J}_{k+1} = \Pi_\xi(T\tilde{J}_k)$, with weights $\xi = (\xi_1, \xi_2)$:

$$r_{k+1} = \arg\min_r \Big[\xi_1\big(r-(T\tilde{J}_k)(1)\big)^2 + \xi_2\big(2r-(T\tilde{J}_k)(2)\big)^2\Big]$$

- With straightforward calculation

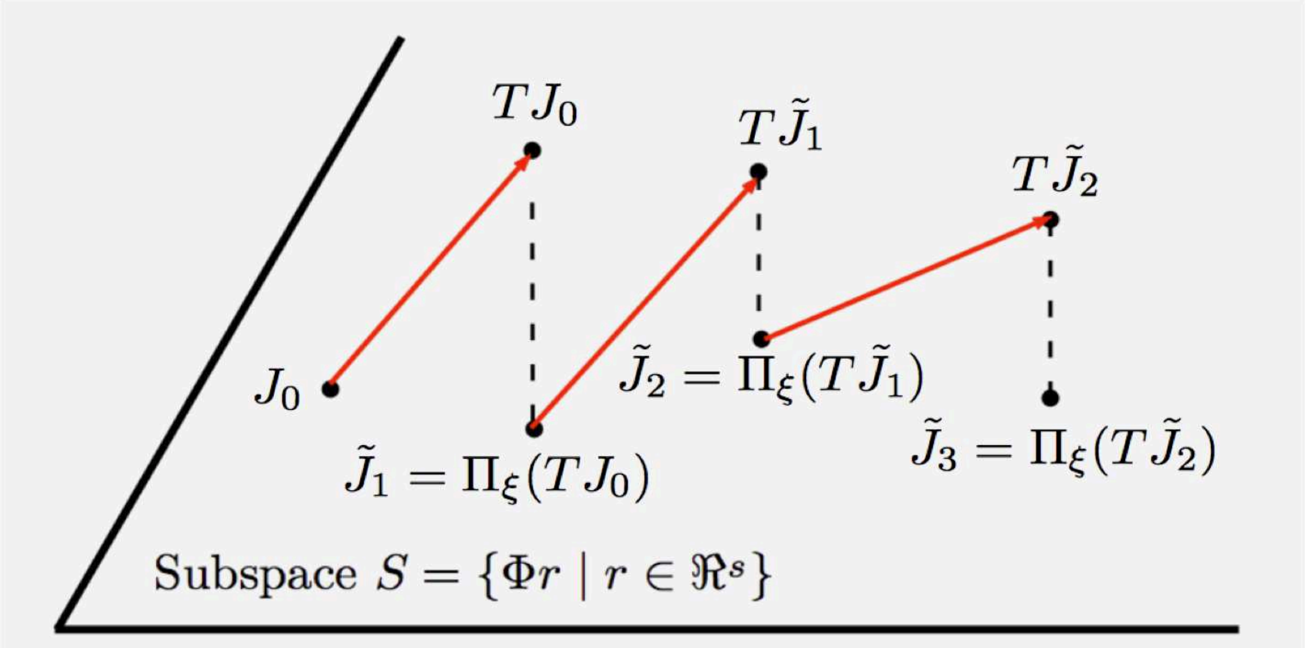
$$r_{k+1} = \alpha\beta r_k, \qquad \text{where } \beta = 2(\xi_1+2\xi_2)/(\xi_1+4\xi_2) > 1$$

- So if $\alpha > 1/\beta$ (e.g., $\xi_1 = \xi_2 = 1$), the sequence $\{r_k\}$ diverges and so does $\{\tilde{J}_k\}$.

- Difficulty is that T is a contraction, but $\Pi_\xi T$ (= least squares fit composed with T) is not.

NORM MISMATCH PROBLEM

- For fitted VI to converge, we need $\Pi_\xi T$ to be a contraction; T being a contraction is not enough



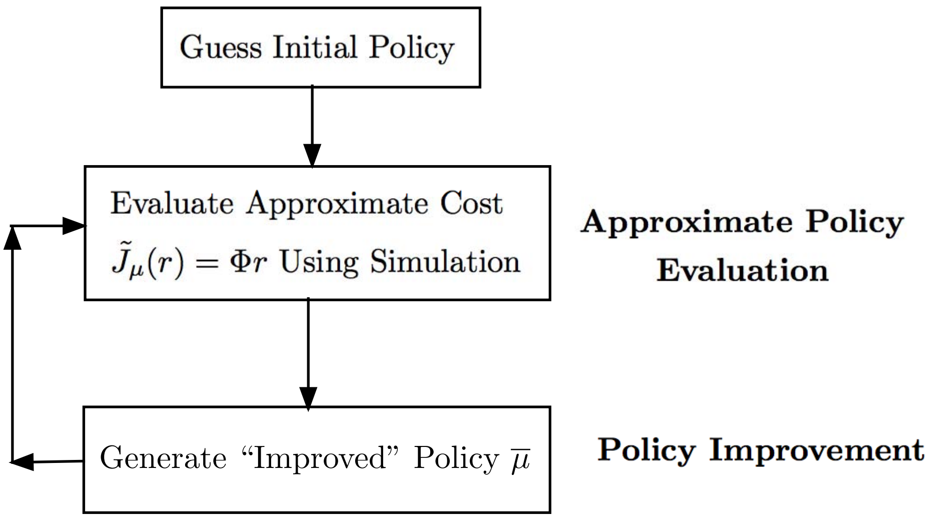
Fitted Value Iteration with Projection

- We need a ξ such that T is a contraction w. r. to the weighted Euclidean norm $\|\cdot\|_\xi$

- Then $\Pi_\xi T$ is a contraction w. r. to $\|\cdot\|_\xi$

- We will come back to this issue, and show how to choose ξ so that $\Pi_\xi T_\mu$ is a contraction for a given μ

APPROXIMATE PI



- Evaluation of typical μ: Linear cost function approximation $\tilde{J}_\mu(r) = \Phi r$, where Φ is full rank $n \times s$ matrix with columns the basis functions, and ith row denoted $\phi(i)'$.

- Policy "improvement" to generate $\overline{\mu}$:

$$\overline{\mu}(i) = \arg \min_{u \in U(i)} \sum_{j=1}^{n} p_{ij}(u)\big(g(i,u,j) + \alpha \phi(j)' r\big)$$

- Error Bound (same as approximate VI): If

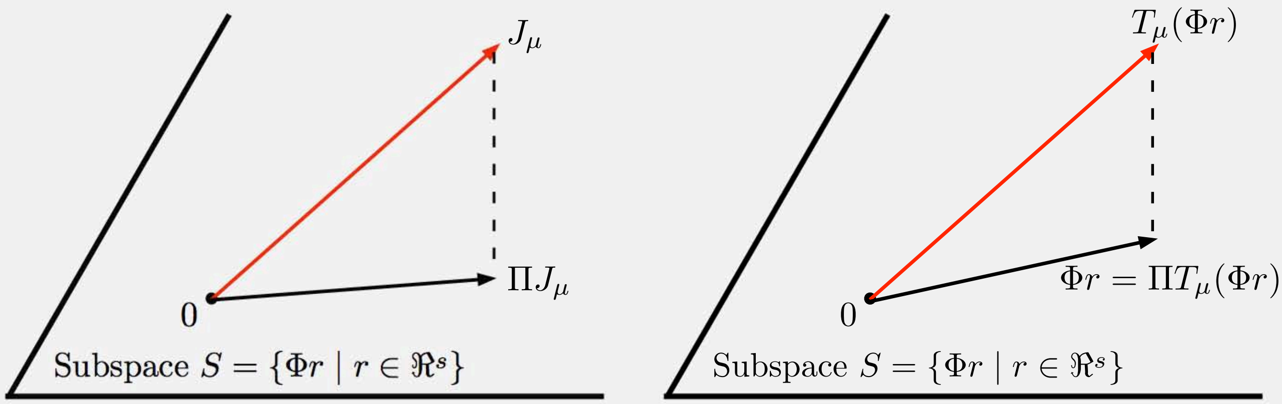
$$\max_i |\tilde{J}_{\mu^k}(i, r_k) - J_{\mu^k}(i)| \leq \delta, \qquad k = 0, 1, \ldots$$

the sequence $\{\mu^k\}$ satisfies

$$\limsup_{k \to \infty} \max_i \big(J_{\mu^k}(i) - J^*(i)\big) \leq \frac{2\alpha\delta}{(1-\alpha)^2}$$

APPROXIMATE POLICY EVALUATION

- Consider approximate evaluation of J_μ, the cost of the current policy μ by using simulation.
  - Direct policy evaluation - generate cost samples by simulation, and optimization by least squares
  - Indirect policy evaluation - solving the projected equation $\Phi r = \Pi T_\mu(\Phi r)$ where Π is projection w/ respect to a suitable weighted Euclidean norm

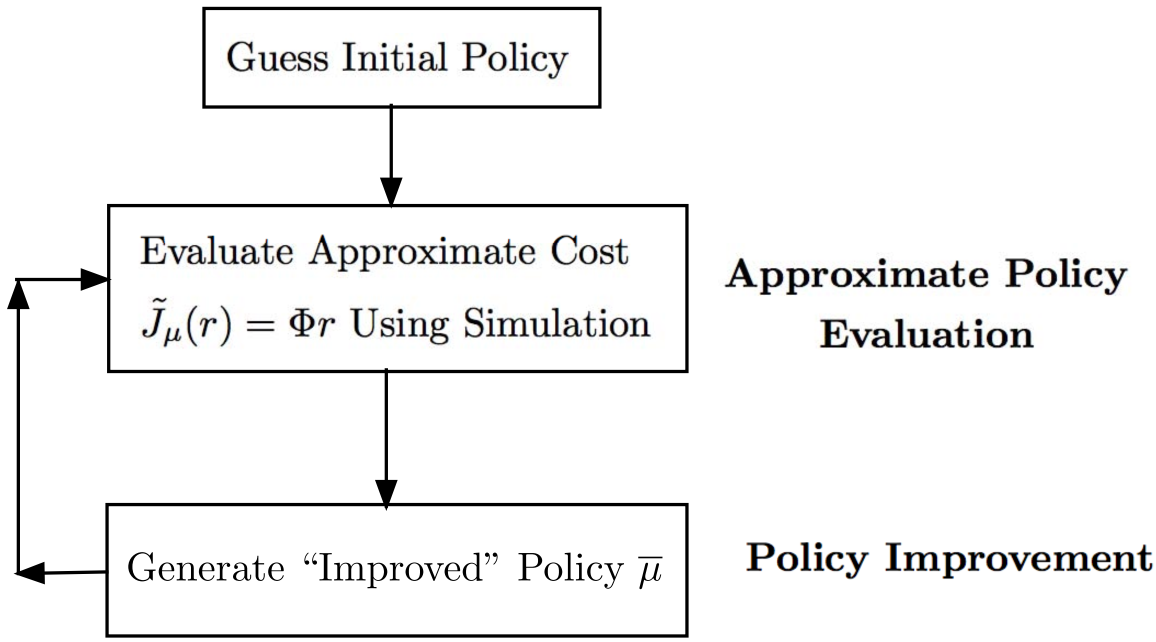


Direct Method: Projection of cost vector J_μ

Indirect Method: Solving a projected form of Bellman's equation

- Recall that projection can be implemented by simulation and least squares

PI WITH INDIRECT POLICY EVALUATION



- Given the current policy μ:
  - We solve the projected Bellman's equation
  $$\Phi r = \Pi T_\mu(\Phi r)$$
  - We approximate the solution J_μ of Bellman's equation
  $$J = T_\mu J$$
  with the projected equation solution $\tilde{J}_\mu(r)$

KEY QUESTIONS AND RESULTS

- Does the projected equation have a solution?

- Under what conditions is the mapping ΠT_μ a contraction, so ΠT_μ has unique fixed point?

- Assumption: The Markov chain corresponding to μ has a single recurrent class and no transient states, with steady-state prob. vector ξ, so that

$$\xi_j = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} P(i_k = j \mid i_0 = i) > 0$$

Note that ξ_j is the long-term frequency of state j.

- Proposition: (Norm Matching Property) Assume that the projection Π is with respect to $\|\cdot\|_\xi$, where $\xi = (\xi_1, \ldots, \xi_n)$ is the steady-state probability vector. Then:

  (a) ΠT_μ is contraction of modulus α with respect to $\|\cdot\|_\xi$.

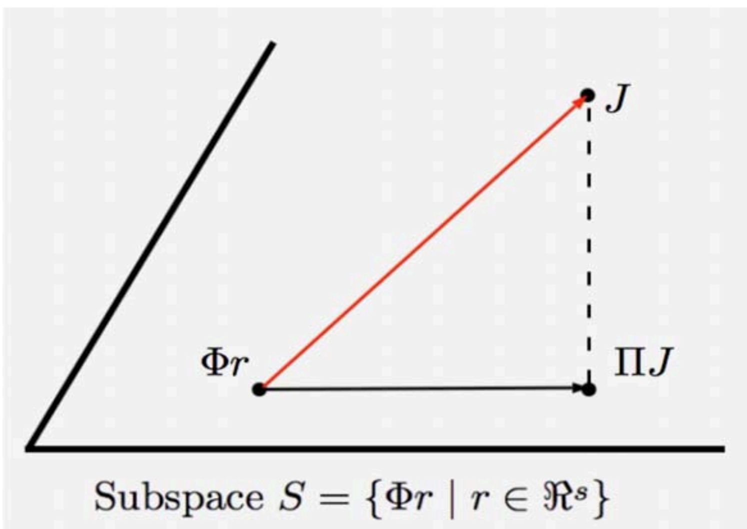
  (b) The unique fixed point Φr^* of ΠT_μ satisfies

$$\|J_\mu - \Phi r^*\|_\xi \leq \frac{1}{\sqrt{1 - \alpha^2}} \|J_\mu - \Pi J_\mu\|_\xi$$

PRELIMINARIES: PROJECTION PROPERTIES

- Important property of the projection Π on S with weighted Euclidean norm $\|\cdot\|_\xi$. For all $J \in \Re^n$, $\Phi r \in S$, the Pythagorean Theorem holds:

$$\|J - \Phi r\|_\xi^2 = \|J - \Pi J\|_\xi^2 + \|\Pi J - \Phi r\|_\xi^2$$



- The Pythagorean Theorem implies that the projection is nonexpansive, i.e.,

$$\|\Pi J - \Pi \bar{J}\|_\xi \le \|J - \bar{J}\|_\xi, \qquad \text{for all } J, \bar{J} \in \Re^n.$$

To see this, note that

$$\begin{aligned} \left\|\Pi(J - \overline{J})\right\|_\xi^2 &\le \left\|\Pi(J - \overline{J})\right\|_\xi^2 + \left\|(I - \Pi)(J - \overline{J})\right\|_\xi^2 \\ &= \|J - \overline{J}\|_\xi^2 \end{aligned}$$

PROOF OF CONTRACTION PROPERTY

- Lemma: If P is the transition matrix of μ,

$$\|Pz\|_\xi \leq \|z\|_\xi, \qquad z \in \Re^n,$$

where ξ is the steady-state prob. vector.
Proof: For all $z \in \Re^n$

$$\|Pz\|_\xi^2 = \sum_{i=1}^n \xi_i \left(\sum_{j=1}^n p_{ij} z_j \right)^2 \leq \sum_{i=1}^n \xi_i \sum_{j=1}^n p_{ij} z_j^2$$
$$= \sum_{j=1}^n \sum_{i=1}^n \xi_i p_{ij} z_j^2 = \sum_{j=1}^n \xi_j z_j^2 = \|z\|_\xi^2.$$

The inequality follows from the convexity of the quadratic function, and the next to last equality follows from the defining property $\sum_{i=1}^n \xi_i p_{ij} = \xi_j$

- Using the lemma, the nonexpansiveness of Π, and the definition $T_\mu J = g + \alpha P J$, we have

$$\|\Pi T_\mu J - \Pi T_\mu \bar{J}\|_\xi \leq \|T_\mu J - T_\mu \bar{J}\|_\xi = \alpha \|P(J - \bar{J})\|_\xi \leq \alpha \|J - \bar{J}\|_\xi$$

for all $J, \bar{J} \in \Re^n$. Hence ΠT_μ is a contraction of modulus α.

PROOF OF ERROR BOUND

- Let Φr^* be the fixed point of ΠT. We have

$$\|J_\mu - \Phi r^*\|_\xi \leq \frac{1}{\sqrt{1-\alpha^2}} \|J_\mu - \Pi J_\mu\|_\xi.$$

Proof: We have

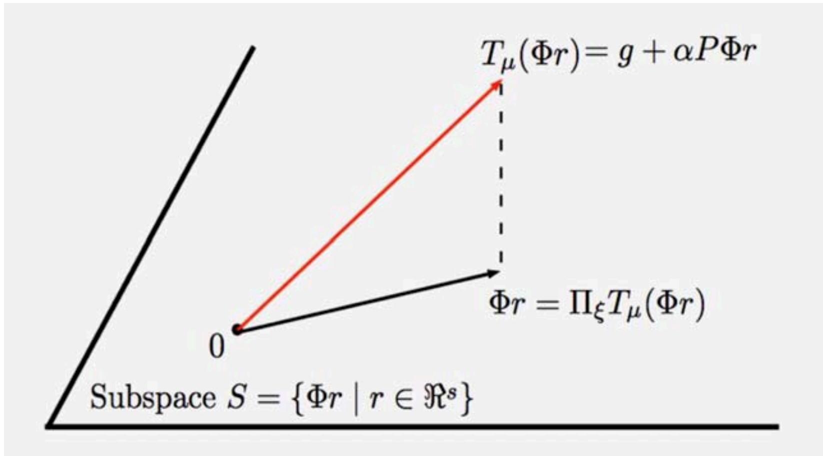
$$\begin{aligned}
\|J_\mu - \Phi r^*\|_\xi^2 &= \|J_\mu - \Pi J_\mu\|_\xi^2 + \left\|\Pi J_\mu - \Phi r^*\right\|_\xi^2 \\
&= \|J_\mu - \Pi J_\mu\|_\xi^2 + \left\|\Pi T J_\mu - \Pi T(\Phi r^*)\right\|_\xi^2 \\
&\leq \|J_\mu - \Pi J_\mu\|_\xi^2 + \alpha^2 \|J_\mu - \Phi r^*\|_\xi^2,
\end{aligned}$$

where

- The first equality uses the Pythagorean Theorem
- The second equality holds because J_μ is the fixed point of T and Φr^* is the fixed point of ΠT
- The inequality uses the contraction property of ΠT.

Q.E.D.

MATRIX FORM OF PROJECTED EQUATION



- The solution Φr^* satisfies the orthogonality condition: The error

$$\Phi r^* - (g + \alpha P \Phi r^*)$$

is "orthogonal" to the subspace spanned by the columns of Φ.

- This is written as

$$\Phi' \Xi \big(\Phi r^* - (g + \alpha P \Phi r^*) \big) = 0,$$

where Ξ is the diagonal matrix with the steady-state probabilities $\xi_1, \ldots, \xi_n$ along the diagonal.

- Equivalently, $C r^* = d$, where

$$C = \Phi' \Xi (I - \alpha P) \Phi, \qquad d = \Phi' \Xi g$$

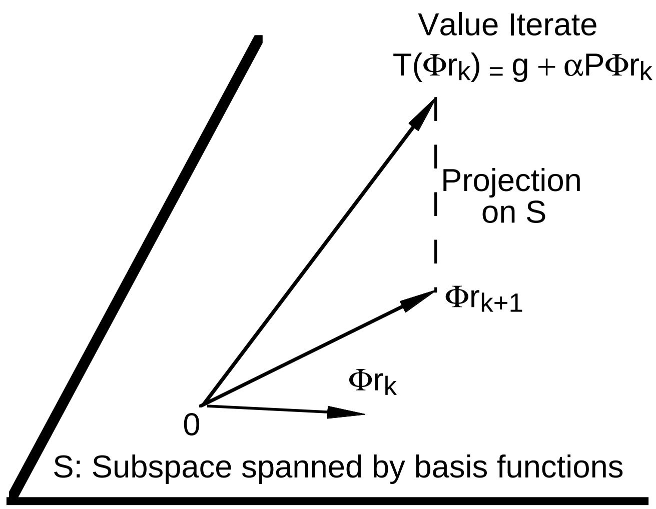
but computing C and d is HARD (high-dimensional inner products).

SOLUTION OF PROJECTED EQUATION

- Solve $Cr^* = d$ by matrix inversion: $r^* = C^{-1}d$
- Alternative: Projected Value Iteration (PVI)

$$\Phi r_{k+1} = \Pi T(\Phi r_k) = \Pi(g + \alpha P \Phi r_k)$$

Converges to r^* because ΠT is a contraction.



- PVI can be written as:

$$r_{k+1} = \arg\min_{r \in \Re^s} \left\| \Phi r - (g + \alpha P \Phi r_k) \right\|_\xi^2$$

By setting to 0 the gradient with respect to r,

$$\Phi' \Xi \big(\Phi r_{k+1} - (g + \alpha P \Phi r_k) \big) = 0,$$

which yields

$$r_{k+1} = r_k - (\Phi' \Xi \Phi)^{-1} (C r_k - d)$$

SIMULATION-BASED IMPLEMENTATIONS

- Key idea: Calculate simulation-based approximations based on k samples

$$C_k \approx C, \qquad d_k \approx d$$

- Approximate matrix inversion $r^* = C^{-1}d$ by

$$\hat{r}_k = C_k^{-1} d_k$$

This is the LSTD (Least Squares Temporal Differences) method.

- PVI method $r_{k+1} = r_k - (\Phi'\Xi\Phi)^{-1}(Cr_k - d)$ is approximated by

$$r_{k+1} = r_k - G_k(C_k r_k - d_k)$$

where

$$G_k \approx (\Phi'\Xi\Phi)^{-1}$$

This is the LSPE (Least Squares Policy Evaluation) method.

- Key fact: C_k, d_k, and G_k can be computed with low-dimensional linear algebra (of order s; the number of basis functions).

SIMULATION MECHANICS

- We generate an infinitely long trajectory $(i_0, i_1, \ldots)$ of the Markov chain, so states i and transitions (i, j) appear with long-term frequencies ξ_i and p_{ij}.

- After generating each transition (i_t, i_{t+1}), we compute the row $\phi(i_t)'$ of Φ and the cost component $g(i_t, i_{t+1})$.

- We form

$$d_k = \frac{1}{k+1} \sum_{t=0}^{k} \phi(i_t) g(i_t, i_{t+1}) \approx \sum_{i,j} \xi_i p_{ij} \phi(i) g(i, j) = \Phi' \Xi g = d$$

$$C_k = \frac{1}{k+1} \sum_{t=0}^{k} \phi(i_t) \big(\phi(i_t) - \alpha \phi(i_{t+1})\big)' \approx \Phi' \Xi (I - \alpha P) \Phi = C$$

Also in the case of LSPE

$$G_k = \frac{1}{k+1} \sum_{t=0}^{k} \phi(i_t) \phi(i_t)' \approx \Phi' \Xi \Phi$$

- Convergence based on law of large numbers.

- C_k, d_k, and G_k can be formed incrementally. Also can be written using the formalism of temporal differences (this is just a matter of style)

OPTIMISTIC VERSIONS

- Instead of calculating nearly exact approximations $C_k \approx C$ and $d_k \approx d$, we do a less accurate approximation, based on few simulation samples
- Evaluate (coarsely) current policy μ, then do a policy improvement
- This often leads to faster computation (as optimistic methods often do)
- Very complex behavior (see the subsequent discussion on oscillations)
- The matrix inversion/LSTD method has serious problems due to large simulation noise (because of limited sampling) - particularly if the C matrix is ill-conditioned
- LSPE tends to cope better because of its iterative nature (this is true of other iterative methods as well)
- A stepsize $\gamma \in (0, 1]$ in LSPE may be useful to damp the effect of simulation noise

$$r_{k+1} = r_k - \gamma G_k(C_k r_k - d_k)$$

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