MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.241: Dynamic Systems—Spring 2011

HOMEWORK 5 SOLUTIONS

Exercise 7.2 a) Suppose $c = 2$. Then the impulse response of the system is

$$h(t) = 2(e^{-t} - e^{-2t}) \quad \text{for } \ \mathrm{t} \geq 0$$

One may assume that $u(t) = 0$ for $t < 0$ this will just alter the lower limit of integration in the convolution formula, but will not affect the state space description, note also that the system is causal.

$$y(t) = \int_0^t 2(e^{-(t-\tau)} - e^{-2(t-\tau)})u(\tau)d\tau \quad \mathrm{t} \geq 0.$$

Hence by use of Leibniz differentiation rule,

$$\dot{y}(t) = 2\int_0^t \frac{d}{dt}(e^{-(t-\tau)} - e^{-2(t-\tau)})u(\tau)d\tau = 2\int_0^t (2e^{-2(t-\tau)} - e^{-(t-\tau)})u(\tau)d\tau,$$

and

$$\ddot{y}(t) = 2\int_0^t (e^{-(t-\tau)} - 4e^{-2(t-\tau)})u(\tau)d\tau + 2u(t).$$

Now, let $x_1(t) = y(t)$ and $x_2(t) = \dot{y}(t)$, then we have

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} u(t).$$

Since $x_1(t)$ and $x_2(t)$ can be written as $\dot{x} = Ax + Bu$, there variables satisfy the continuous time state property and are this valid state variables.
b) The transfer function of the system is

$$H(s) = \frac{2(s+2) - c(s+1)}{s^2 + 3s + 2}, \quad Re(s) > -1.$$

When $c = 2$ there are no s terms in the numerator, which implies that the output $y(t)$ only depends on $u(t)$ but not on $\dot{u}(t)$. Our selection of state variables is valid only for $c = 2$. If $c \neq 2$, the reachability canonical form may guide us to the selection of state variables

Exercise 7.3 In this problem, we have

$$\dot{y} = -a_0(t)y(t) + b_0(t)u(t) + b_1(t)\dot{u}(t).$$

That is,

$$y = \int -a_0(t)y(t) + b_0(t)u(t) + \int b_1(t)\dot{u}(t).$$

Notice that in the TI case, the coefficients a_0, b_0, and b_1 were constants, so we were able to integrate the term $\int b_1\dot{u}(t)$. In this case, we can still get rid of the $\dot{u}(t)$, by integration by parts. We have

$$\int b_1(t)\dot{u}(t)\mathrm{d}t = b_1(t)u(t) - \int u(t)\dot{b_1}(t)\mathrm{d}t.$$

So, our equation becomes

$$y = b_1(t)u(t) + \int -a_0(t)y(t) + (b_0(t) - \dot{b_1}(t))u(t).$$

Now, let

$$\dot{x} = -a_0(t)y(t) + (b_0(t) - \dot{b_1}(t))u(t),$$

we have that

$$y = x + b_1(t)u(t),$$

and substituting y in the equation for $\dot{x}$ we get:

$$\begin{aligned} \dot{x} &= -a_0(t)x(t) + (b_0(t) - \dot{b_1}(t) - a_0(t)b_1(t))u(t) \\ y &= x + b_1(t)u(t) \end{aligned}$$

Exercise 10.1 a) $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

b) Let $J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_q \end{bmatrix}$ be the Jordan form decomposition of A, $A = MJM^{-1}$.
Note that $J^k = 0 \Leftrightarrow J_i^k = 0, 1 \leq i \leq q$.
Also note that $A^k = MJ^kM^{-1}$ and hence $A^k = 0 \Leftrightarrow J^k = 0$.
Thus, it suffices to show that $J_i^k = 0$, for all $i \in \{1, \ldots, q\}$ for some finite positive power k iff all the eigenvalues of A are 0.
First, we prove sufficiency: If all the eigenvalues of A are 0, then the corresponding Jordan blocks have zero diagonal elements and are such that $J_i^{n_i} = 0$ for every i, where n_i is the size of J_i. Let $k_o = \max_{1 \leq i \leq q} n_i$. k_o is finite and we have $J_i^{k_o} = 0, 1 \leq i \leq q$.
Next, we proof necessity. Suppose there exists at least one eigenvalue of A, say λ_{i_o}, that is non-zero. Note that the diagonal elements of the k^{th} power of the corresponding Jordan block(s) are $\lambda_{i_o}^k$, for any positive power k. Hence, there exists at least one i such that $J_i^k \neq 0$, for any positive power k. If A has size n, then the size of each of the Jordan blocks in its Jordan form decomposition is at most n. Hence $k_o \leq n$ and $A^n = 0$.

c) The smallest value is k_o defined in part (b). Here, $k_o = n_q$.

d) Let $J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_q \end{bmatrix}$ be the Jordan form decomposition of A. We have that:

$$\mathcal{R}(A^{k+1}) = \mathcal{R}(A^k) \Leftrightarrow \mathcal{R}(J^{k+1}) = \mathcal{R}(J^k)$$

Thus it suffices to look for the smallest value of k for which $\mathcal{R}(J^{k+1}) = \mathcal{R}(J^k)$. Note that a Jordan block associated with a non-zero eigenvalue has full column rank, and retains full column rank when raised to any positive power k. On the other hand, a nilpotent Jordan block of size n_i has column rank $n_i - 1$, and is such that $rank J_i^k = \max\{0, n_i - k\}$. Let $N = \{i | J_i \text{is nilpotent}\}$. Define $k_{min} = \max_{i \in N} n_i$. k_{min} is the smallest value of k for which $\mathcal{R}(J^{k+1}) = \mathcal{R}(J^k)$.

Exercise 11.1.

Since the characteristic polynomial of A is a determinant of a matrix $zI - A$,

$$det(zI - A) = det((zI - A)^T) = det(zI - A^T),$$

first we show that

$$det(zI - A_1) = det(zI - A_2) = q(z)$$

for given A_1 and A_2. For

$$A_1 = \begin{pmatrix} -q_{n-1} & 1 & 0 & \cdots & 0 \\ -q_{n-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -q_1 & 0 & 0 & \cdots & 1 \\ -q_0 & 0 & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -q_0 & -q_1 & -q_2 & \cdots & -q_{n-1} \end{pmatrix},$$

we have

$$zI - A_1 = \begin{pmatrix} z + q_{n-1} & -1 & 0 & \cdots & 0 \\ q_{n-2} & z & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_1 & 0 & 0 & \cdots & -1 \\ q_0 & 0 & 0 & \cdots & z \end{pmatrix} \quad \text{and} \quad zI - A_2 = \begin{pmatrix} z & -1 & 0 & \cdots & 0 \\ 0 & z & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \\ q_0 & q_1 & q_2 & \cdots & z + q_{n-1} \end{pmatrix}.$$

Recall that $det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$ for

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix},$$

where A_{ij} is a cofactor matrix corresponding a_{ij}. Then,

$$
\begin{aligned}
det(zI-A_1) &= (z+q_{n-1})\begin{vmatrix} z & -1 & 0 & \cdots & 0 & 0 \\ 0 & z & -1 & \cdots & 0 & 0 \\ 0 & 0 & z & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & z & -1 \\ 0 & 0 & 0 & \cdots & 0 & z \end{vmatrix} - q_{n-2}\begin{vmatrix} -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & z & -1 & \cdots & 0 & 0 \\ 0 & 0 & z & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & z & -1 \\ 0 & 0 & 0 & \cdots & 0 & z \end{vmatrix} \\
&+ \quad q_{n-3}\begin{vmatrix} -1 & 0 & 0 & \cdots & 0 & 0 \\ z & -1 & 0\ldots & 0 & 0 & \\ 0 & 0 & z & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & z & -1 \\ 0 & 0 & 0 & \cdots & 0 & z \end{vmatrix} - \cdots \pm q_0 \begin{vmatrix} -1 & 0 & 0 & \cdots & 0 & 0 \\ z & -1 & 0 & \cdots & 0 & 0 \\ 0 & z & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & 0 & \cdots & z & -1 \end{vmatrix},
\end{aligned}
$$

where the last $\pm$ depends on whether n is an even or odd number. Similarly if we take the determinant of $zI - A_2$ using cofactors on the last row of $zI - A_2$ it is clear that we have

$$det(zI - A_1) = det(zI - A_2) = q(z).$$

Also it is true that

$$det(zI - A) = det((zI - A)^T) = det(zI - A^T).$$

Hence

$$det(zI - A_1) = det(zI - A_1^T) = det(zI - A_2) = det(zI - A_2^T) = q(z).$$

Then we have

$$
\begin{aligned}
q(z) &= (z + q_{n-1})z^{n-1} + q_{n-2}z^{n-2} + \cdots + q_1 z + q_0 \\
\therefore q(z) &= z^n + q_{n-1}z^{n-1} + q_{n-2}z^{n-2} + \cdots + q_1 z + q_0.
\end{aligned}
$$

b) For A_2, we have

$$
\lambda_i I - A_2 = \begin{pmatrix} \lambda_i & -1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_i & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i & -1 \\ q_0 & q_1 & q_3 & \cdots & q_{n-2} & \lambda_i + q_{n-1} \end{pmatrix}
$$

Suppose v_1 is an eigenvector corresponding to λ_i, then $v_i \in \mathcal{N}(\lambda_i I - A_2)$, i.e.,

$$
\begin{pmatrix} \lambda_i & -1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_i & -1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i & -1 \\ q_0 & q_1 & q_3 & \cdots & q_{n-2} & \lambda_i + q_{n-1} \end{pmatrix} \underline{v}_i = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}. \qquad (1)
$$

It is clear from Eqn 1 that the only nonzero $\underline{v}_i$ is

$$\underline{v}_i = \begin{pmatrix} 1 \\ \lambda_i \\ \lambda_i^2 \\ \vdots \\ \lambda_i^{n-1} \end{pmatrix}.$$

Then Eqn 1 becomes

$$\begin{pmatrix} \lambda_i \\ 0 \\ \vdots \\ q_0 \end{pmatrix} + \begin{pmatrix} -\lambda_i \\ \lambda_i^2 \\ \vdots \\ \lambda_i q_1 \end{pmatrix} + \cdots + \begin{pmatrix} 0 \\ \vdots \\ -\lambda_i^{n-1} \\ (\lambda_i + q_{n-1})\lambda_i^{n-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ q_0 + q_1\lambda_i + \cdots + \lambda_i^{n-1} q_{n-1} + \lambda_i^n \end{pmatrix} = \underline{0}$$

since λ_i is a root of $q(\lambda)$.

c) Consider

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & 5 & -2 \end{pmatrix}.$$

Its eigenvalues are $\lambda_1 = -1$, $\lambda_2 = -3$, and $\lambda_3 = 2$. Note that this A has the form of A_2 thus the corresponding eigenvectors can be written as follows:

$$\underline{v}_1 = \begin{pmatrix} 1 \\ \lambda_1 \\ \lambda_1^2 \end{pmatrix}, \ \underline{v}_2 = \begin{pmatrix} 1 \\ \lambda_2 \\ \lambda_2^2 \end{pmatrix}, \ \underline{v}_3 = \begin{pmatrix} 1 \\ \lambda_3 \\ \lambda_3^2 \end{pmatrix}.$$

Using those three eigenvectors we can obtain the similarity transformation matrix, M, to make A diagonal:

$$M = \begin{pmatrix} | & | & | \\ \underline{v}_1 & \underline{v}_2 & \underline{v}_3 \\ | & | & | \end{pmatrix}.$$

Thus with

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

we have

$$A = M\Lambda M^{-1},$$

which implies that

$$
\begin{aligned}
A^k &= M\Lambda^k M^{-1} = M\begin{pmatrix} \lambda_1^k & 0 & 0 \\ 0 & \lambda_2^k & 0 \\ 0 & 0 & \lambda_3^k \end{pmatrix} M^{-1} \\
&= \begin{pmatrix} 1 & 1 & 1 \\ -1 & -3 & 2 \\ 1 & 9 & 4 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & -\frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{5} & -\frac{1}{10} & \frac{1}{10} \\ \frac{1}{5} & \frac{4}{v15} & \frac{1}{15} \end{pmatrix},
\end{aligned}
$$

and

$$
\begin{aligned}
e^{At} &= M\begin{pmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{pmatrix} M^{-1} \\
&= \begin{pmatrix} 1 & 1 & 1 \\ -1 & -3 & 2 \\ 1 & 9 & 4 \end{pmatrix}\begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-3t} & 0 \\ 0 & 0 & e^{2t} \end{pmatrix}\begin{pmatrix} 1 & -\frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{5} & -\frac{1}{10} & \frac{1}{10} \\ \frac{1}{5} & \frac{4}{15} & \frac{1}{15} \end{pmatrix}.
\end{aligned}
$$

Exercise 11.3 This equality can be shown in a number of ways. Here we will see two. One way is by diagonalization of the matrix

$$
\begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix}.
$$

Its characteristic equation is $\chi(A) = (\lambda - \sigma)^2 + \omega^2$, yielding eigenvalues $\lambda = \sigma \pm j\omega$. Using the associated eigenvectors, we can show that it has diagonalization

$$
\begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -j & j \end{pmatrix}\begin{pmatrix} \sigma - j\omega & \\ & \sigma + j\omega \end{pmatrix}\begin{pmatrix} \frac{1}{2} & j\frac{1}{2} \\ \frac{1}{2} & -j\frac{1}{2} \end{pmatrix}.
$$

Now

$$
\begin{aligned}
\exp\left[t\begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix}\right] &= \begin{pmatrix} 1 & 1 \\ -j & j \end{pmatrix}\begin{pmatrix} e^{\sigma}e^{-j\omega t} & \\ & e^{\sigma}e^{j\omega t} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & j\frac{1}{2} \\ \frac{1}{2} & -j\frac{1}{2} \end{pmatrix} \\
&= \begin{pmatrix} e^{\sigma}\frac{e^{j\omega t}+e^{-j\omega t}}{2} & e^{\sigma}\frac{e^{j\omega t}-e^{-j\omega t}}{j2} \\ e^{\sigma}\frac{-e^{j\omega t}+e^{-j\omega t}}{j2} & e^{\sigma}\frac{e^{j\omega t}+e^{-j\omega t}}{2} \end{pmatrix} \\
&= \begin{pmatrix} e^{\sigma}\cos(\omega t) & e^{\sigma}\sin(\omega t) \\ -e^{\sigma}\sin(\omega t) & e^{\sigma}\cos(\omega t) \end{pmatrix}.
\end{aligned}
$$

An arguably simpler way to achieve this result is by applying the inverse Laplace transform identity $e^{tA} = \mathcal{L}^{-1}\left[(sI - A)^{-1}\right]$. We have

$$
(sI - A) = \begin{pmatrix} s - \sigma & -\omega \\ \omega & s - \sigma \end{pmatrix}
$$

and so

$$
(sI - A)^{-1} = \frac{1}{(s-\sigma)^2 + \omega^2}\begin{pmatrix} s - \sigma & \omega \\ -\omega & s - \sigma \end{pmatrix}.
$$

Taking the inverse Laplace transform element-wise gives us the previous result.

Exercise 11.4 This equality is shown through the definition of the matrix exponential. The derivation is as follows.

$$\begin{aligned}\exp\left[t\begin{pmatrix} A & \\ & B\end{pmatrix}\right] &= \sum_{k=0}^{\infty}\frac{1}{k!}t^k\begin{pmatrix} A & \\ & B\end{pmatrix}^k \\ &= \sum_{k=0}^{\infty}\frac{1}{k!}\begin{pmatrix} t^kA^k & \\ & t^kB^k\end{pmatrix} \\ &= \begin{pmatrix} \sum_{k=0}^{\infty}\frac{1}{k!}t^kA^k & \\ & \sum_{k=0}^{\infty}\frac{1}{k!}t^kB^k\end{pmatrix} = \begin{pmatrix} e^{tA} & \\ & e^{tB}\end{pmatrix}\end{aligned}$$

Exercise 11.5 By direct substitution for the proposed solution, we have:

$$\begin{aligned}e^{-tA}Be^{tA}x(t) &= e^{-tA}Be^{tA}e^{-tA}e^{(t-t_o)(A+B)}e^{t_oA}x(t_o) \\ &= e^{-tA}Be^{(t-t_o)(A+B)}e^{t_oA}x(t_o) \qquad (2)\end{aligned}$$

Differentiating the proposed solution, we have:

$$\frac{dx(t)}{dt} = -Ae^{-tA}e^{(t-t_o)(A+B)}e^{t_oA}x(t_o) + e^{-tA}(A+B)e^{(t-t_o)(A+B)}e^{t_oA}x(t_o)$$

But since $Ae^{-tA} = e^{-tA}A$, this is:

$$\begin{aligned}&= e^{-tA}(-A+A+B)e^{(t-t_o)(A+B)}e^{t_oA}x(t_o) \\ &= e^{-tA}Be^{(t-t_o)(A+B)}e^{t_oA}x(t_o) \qquad (3)\end{aligned}$$

Since (3) and (2) are equal, the proposed solution satisfies the system of ODEs and hence is a solution. Moreover, it can be shown that the solution is unique (though this is not the subject of this class).