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**6.245: MULTIVARIABLE CONTROL SYSTEMS**

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# Kalman-Yakubovich-Popov Lemma [1]

A simplified version of KYP lemma was used earlier in the derivation of optimal H2 controller, where it states existence of a stabilizing solution of a Riccati equation associated with a non-singular abstract H2 optimization problem. This lecture presents the other side of the KYP lemma: necessary and sufficient conditions of positive semidefiniteness of quadratic integrals subject to LTI constraints.

## 6.1 KYP Lemma on Positivity

Just as in the simplified version of KYP Lemma, consider the LTI state space equations

$$\dot{p} = ap + bq,\ p(0) = p_0,\ \lim_{t\to\infty} p(t) = 0, \tag{6.1}$$

where $a, b$ are given real matrices of dimensions $n$-by-$n$ and $n$-by-$m$ respectively. In addition, consider a Hermitian form $\sigma :\ \mathbf{C}^n \times \mathbf{C}^m \mapsto \mathbf{R}$:

$$\sigma :\quad \sigma(\bar{p}, \bar{q}) = \begin{bmatrix} \bar{p} \\ \bar{q} \end{bmatrix}' \Sigma \begin{bmatrix} \bar{p} \\ \bar{q} \end{bmatrix}, \tag{6.2}$$

where

$$\Sigma = \Sigma' = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix},$$



[1] Version of March 1, 2004

is a given symmetric matrix (i.e. a Hermitian matrix with *real* coefficients). In the simplified version, $\Sigma$ was positive semidefinite, which is equivalent to $\sigma(p,q)$ being representable in the form

$$\sigma(\bar{p},\bar{q}) = |c\bar{p} + d\bar{q}|^2.$$

In this lecture, the case of interest is when $\sigma$ is indefinite, and we are looking for constructively verifiable conditions which guarantee that the integral

$$\int_0^\infty \sigma(p(t),q(t))dt \tag{6.3}$$

remains bounded from below subject to (6.1) for all fixed $p_0$. In addition, a related (but not equivalent) question of whether the matrix

$$\Pi(j\omega) = \begin{bmatrix} (j\omega I - a)^{-1}b \\ I \end{bmatrix}' \Sigma \begin{bmatrix} (j\omega I - a)^{-1}b \\ I \end{bmatrix} \tag{6.4}$$

is positive semidefinite for all $\omega \in \mathbf{R}$ such that $\det(j\omega I - a) \neq 0$ will be studied. Finally, existence of a symmetric matrix æ = æ′ such that

$$\sigma(\bar{p},\bar{q}) + 2\mathrm{Re}\ p'\text{æ}(ap + bq) \geq 0 \quad \forall\ \bar{p} \in \mathbf{C}^n, \bar{q} \in \mathbf{C}^m \tag{6.5}$$

turns out to be very closely related to the positivity conditions mentioned above.

### 6.1.1 Formulation of a KYP Lemma

The statements of the formulation given below should be complemented by the statements of the simplified version of the KYP Lemma given earlier. First, let us define several conditions, each of which may be true or false for a particular choice of the coefficients $a, b, \Sigma$:

- (T) the integral in (6.3) remains bounded from below subject to (6.1) for $p_0 = 0$;
- (T+) the integral in (6.3) remains bounded from below subject to (6.1) for all $p_0$;
- (F) matrix $\Pi(j\omega)$ from (6.4) is positive semidefinite for all $\omega \in \mathbf{R}$ such that $\det(j\omega I - a) \neq 0$;
- (F+) matrix $\Pi(j\omega)$ from (6.4) is positive definite for all $\omega \in \mathbf{R}$ except, possibly, a finite number of frequencies;
- (Q) there exists matrix æ = æ′ such that (6.5) holds.

**Theorem 6.1** *Assume that the pair $(a, b)$ is stabilizable. Then*

*(a) (T) is equivalent to (F);*

*(b) (T+) is equivalent to (Q);*

*(c) (F+) implies (Q);*

*(d) when $(a, b)$ is controllable, $F$ implies $Q$.*

### 6.1.2 KYP Lemma and Hamiltonian Matrrices

When $\Sigma_{22}$ is invertible, a Hamiltonian matrix $\mathcal{H}$ can be associated with the 3-typle $(a, b, \Sigma)$ according to

$$\mathcal{H} = \mathcal{H}[a, b, \Sigma] = \left[ \begin{array}{cc} a - b\Sigma_{22}^{-1}\Sigma_{21} & b\Sigma_{22}^{-1}b' \\ \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} & -a' + \Sigma_{12}\Sigma_{22}^{-1}b' \end{array} \right]. \tag{6.6}$$

When a coordinate change

$$p = S_{11}p^+, \quad q = S_{12}p^+ + S_{22}q^+, \quad \det(S_{ii}) \neq 0,$$

is applied to the signal variables in (6.1),(6.2), this causes modification of $a, b, \Sigma$, and, hence, of $\mathcal{H}$. In addition, replacing $\sigma(p, q)$ by

$$\sigma^+(p, q) + 2\mathrm{Re} p' æ(ap + bq)$$

also leads to a modified $\Sigma$, and, hence, $\mathcal{H}$. Let us call two 3-typles $(a_1, b_1, \Sigma_1)$ and $(a_2, b_2, \Sigma_2)$ *similar* if one can be obtained from another by applying a series of the such transformations.

An important addition to the KYP Lemma is given by the following statement.

**Theorem 6.2**

*(a) Equivalent 3-typles $(a, b, \Sigma)$ correspond to* similar *Hamiltonian matrices $\mathcal{H}[a, b, \Sigma]$.*

*(b) If $\jmath\omega$ (with $\omega \in \mathbf{R}$) is an eigenvalue of $\mathcal{H}$ but not an eigenvalue of $a$ then* $\det \Pi(j\omega) = 0$.

Theorem 6.2 allows one to deduct strict positivity versions of (T+), (F+), or $Q$ from the absence of purely imaginary eigenvalues of the corresponding Hamiltonian matrix.

### 6.1.3 Example: H-Infinity Norm Calculation

Let matrices $a, b, c, d$ define a state space model of a stable LTI system. L2 gain of this system does not exceed $\gamma > 0$ if and only if condition (F) (or, equivalently, (T)) holds for

$$\sigma(p, q) = \gamma^2 |q|^2 - |cp + dq|^2.$$

If the pair $(a, b)$ is controllable, according to Theorem 6.1, this is equivalent to existence of a matrix $æ = æ'$ such that

$$\gamma^2 |q|^2 - |cp + dq|^2 + 2\mathrm{Re}\; p' æ (ap + bq) \geq 0.$$

More practically, one can notice that L2 gain of this system is strictly less than $\gamma$ if and only if $\Sigma_{22} = \gamma^2 I - d'd > 0$ and $\Pi(j\omega) > 0$ for all $\omega \in \mathbf{R}$. According to Theorem 6.2, this is equivalent to absense of purely imaginary eigenvalues of the corresponding Hamiltonian matrix. This idea is behind the standard algorithms for calculating H-Infinity norm of a given LTI system.

### 6.1.4 A Simple Case Study

Consider the second order uncontrollable stable LTI system

$$\begin{aligned} \dot{p}_1(t) &= -p_1(t) + q(t), \\ \dot{p}_2(t) &= -p_2(t), \\ v(t) &= -2p_1(t) + p_2(t) + q(t) \end{aligned}$$

with input $q = q(t)$, output $v = v(t)$, and transfer function

$$G(s) = \frac{1 - s}{1 + s}.$$

The L2 gain of this system equals 1. Hence, for $\gamma > 1$, the integral

$$\Phi = \int_0^\infty \{\gamma^2 |q(t)|^2 - |v(t)|^2\} dt$$

is bounded from below for all fixed initial conditions $p_1(0), p_2(0)$, provided that $p(t) \to 0$ as $t \to \infty$. However, for $\gamma = 1$, $\Phi$ is not bounded from below when $p_2(0) \neq 0$, and there exists no 2-by-2 matrix $æ = æ'$ such that

$$|q|^2 - |q + p_2 - 2p_1|^2 + 2 \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}' æ \begin{bmatrix} -p_1 + q \\ -p_2 \end{bmatrix} \geq 0.$$