# 6.252 NONLINEAR PROGRAMMING

# LECTURE 19: DUALITY THEOREMS

## LECTURE OUTLINE

- Duality and L-multipliers (continued)
- Consider the problem

$$\text{minimize } f(x)$$
$$\text{subject to } x \in X, \qquad g_j(x) \le 0, \qquad j = 1, \ldots, r,$$

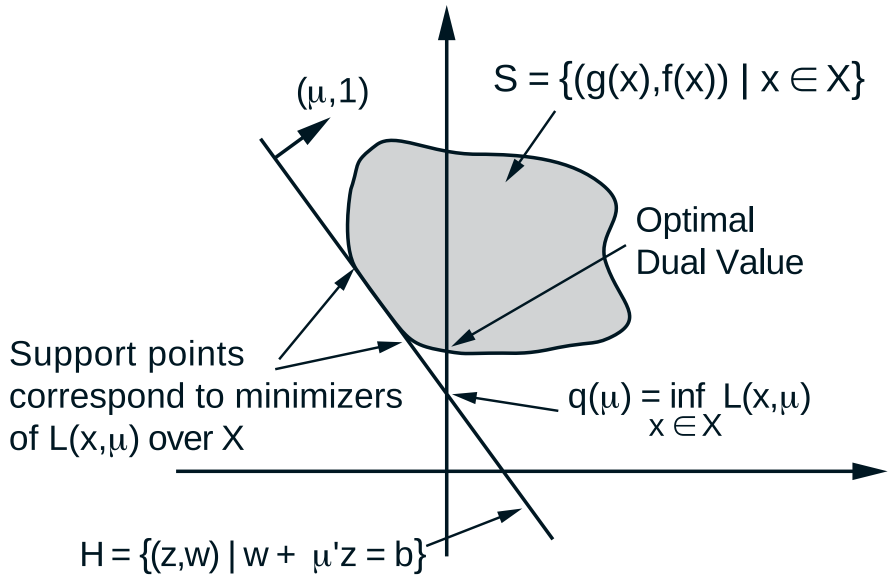
assuming $-\infty < f^* < \infty$.

- $\mu^*$ is a Lagrange multiplier if $\mu^* \ge 0$ and $f^* = \inf_{x \in X} L(x, \mu^*)$.
- The *dual problem* is

$$\text{maximize } q(\mu)$$
$$\text{subject to } \mu \ge 0,$$

where $q$ is the dual function $q(\mu) = \inf_{x \in X} L(x, \mu)$.

# DUAL OPTIMALITY




- Weak Duality Theorem: $q^* \leq f^*$.
- Lagrange Multipliers and Dual Optimal Solutions:

  (a) If there is no duality gap, the set of Lagrange multipliers is equal to the set of optimal dual solutions.

  (b) If there is a duality gap, the set of Lagrange multipliers is empty.

# DUALITY PROPERTIES

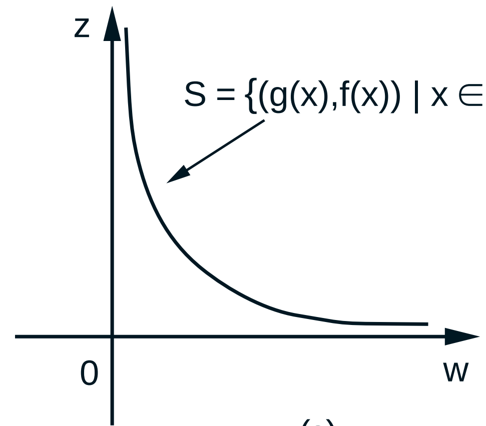
- Optimality Conditions: $(x^*, \mu^*)$ is an optimal solution-Lagrange multiplier pair if and only if

$$
\begin{aligned}
x^* \in X, \quad g(x^*) \leq 0, & \quad \text{(Primal Feasibility)}, \\
\mu^* \geq 0, & \quad \text{(Dual Feasibility)}, \\
x^* = \arg\min_{x \in X} L(x, \mu^*), & \quad \text{(Lagrangian Optimality)}, \\
\mu_j^* g_j(x^*) = 0, \quad j = 1, \ldots, r, & \quad \text{(Compl. Slackness)}.
\end{aligned}
$$

- Saddle Point Theorem: $(x^*, \mu^*)$ is an optimal solution-Lagrange multiplier pair if and only if $x^* \in X$, $\mu^* \geq 0$, and $(x^*, \mu^*)$ is a saddle point of the Lagrangian, in the sense that

$$L(x^*, \mu) \leq L(x^*, \mu^*) \leq L(x, \mu^*), \qquad \forall\, x \in X,\ \mu \geq 0.$$

# INFEASIBLE AND UNBOUNDED PROBLEMS



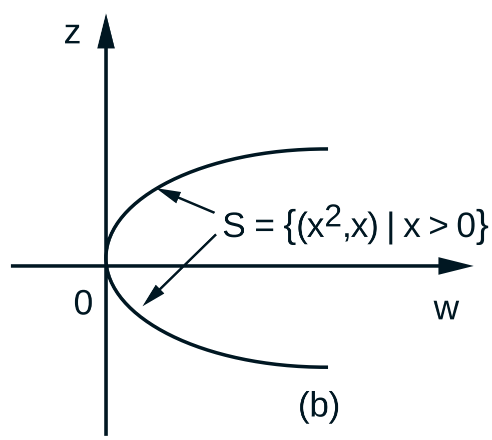

(a)

min $f(x) = 1/x$

s.t. $g(x) = x \le 0$

$x \in X = \{x \mid x > 0\}$

$f^* = \infty, \quad q^* = \infty$



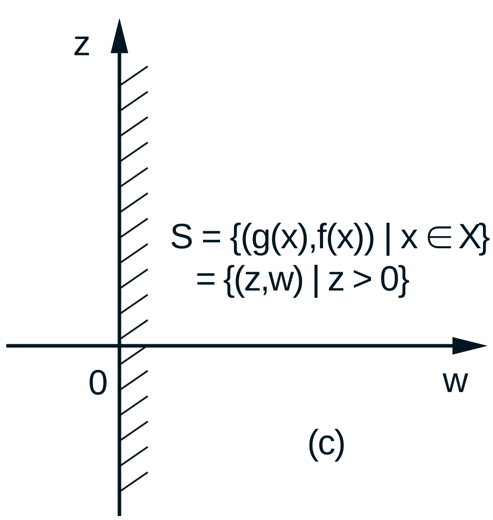

(b)

min $f(x) = x$

s.t. $g(x) = x^2 \le 0$

$x \in X = \{x \mid x > 0\}$

$f^* = \infty, \quad q^* = 0$




(c)

min $f(x) = x_1 + x_2$

s.t. $g(x) = x_1 \le 0$

$x \in X = \{(x_1, x_2) \mid x_1 > 0\}$

$f^* = \infty, \quad q^* = -\infty$

# EXTENSIONS AND APPLICATIONS

- Equality constraints $h_i(x) = 0$, $i = 1, \ldots, m$, can be converted into the two inequality constraints

$$h_i(x) \le 0, \qquad -h_i(x) \le 0.$$

- Separable problems:

$$\begin{aligned} &\text{minimize} \quad \sum_{i=1}^{m} f_i(x_i) \\ &\text{subject to} \quad \sum_{i=1}^{m} g_{ij}(x_i) \le 0, \qquad j = 1, \ldots, r, \\ &\qquad\qquad x_i \in X_i, \qquad i = 1, \ldots, m. \end{aligned}$$

- Separable problem with a single constraint:

$$\begin{aligned} &\text{minimize} \quad \sum_{i=1}^{n} f_i(x_i) \\ &\text{subject to} \quad \sum_{i=1}^{n} x_i \ge A, \quad \alpha_i \le x_i \le \beta_i, \quad \forall\, i. \end{aligned}$$

# DUALITY THEOREM I FOR CONVEX PROBLEMS

- Strong Duality Theorem - Linear Constraints: Assume that the problem

$$\begin{aligned} &\text{minimize} \quad f(x) \\ &\text{subject to} \quad x \in X, \quad a_i'x - b_i = 0, \quad i = 1, \ldots, m, \\ &\qquad\qquad\quad e_j'x - d_j \leq 0, \quad j = 1, \ldots, r, \end{aligned}$$

is feasible and its optimal value $f^*$ is finite. Let also $f$ be convex over $\Re^n$ and let $X$ be polyhedral. Then there exists at least one Lagrange multiplier and there is no duality gap.

- Proof Issues
- Application to Linear Programming

# COUNTEREXAMPLE

- A Convex Problem with a Duality Gap: Consider the two-dimensional problem

$$\text{minimize } f(x)$$
$$\text{subject to } x_1 = 0, \qquad x \in X = \{x \mid x \geq 0\},$$

where

$$f(x) = e^{-\sqrt{x_1 x_2}}, \qquad \forall\, x \in X,$$

and $f(x)$ is arbitrarily defined for $x \notin X$.

- $f$ is convex over $X$ (its Hessian is positive definite in the interior of $X$), and $f^* = 1$.
- Also, for all $\mu \geq 0$ we have

$$q(\mu) = \inf_{x \geq 0} \left\{ e^{-\sqrt{x_1 x_2}} + \mu x_1 \right\} = 0,$$

since the expression in braces is nonnegative for $x \geq 0$ and can approach zero by taking $x_1 \to 0$ and $x_1 x_2 \to \infty$. It follows that $q^* = 0$.

# DUALITY THEOREM II FOR CONVEX PROBLEMS

- Consider the problem

minimize $f(x)$

subject to $x \in X, \qquad g_j(x) \leq 0, \qquad j = 1, \ldots, r.$

- Assume that $X$ is convex and the functions $f : \Re^n \mapsto \Re$, $g_j : \Re^n \mapsto \Re$ are convex over $X$. Furthermore, the optimal value $f^*$ is finite and there exists a vector $\bar{x} \in X$ such that

$$g_j(\bar{x}) < 0, \qquad \forall\, j = 1, \ldots, r.$$

- Strong Duality Theorem: There exists at least one Lagrange multiplier and there is no duality gap.
- Extension to linear equality constraints.