6.262: Discrete Stochastic Processes 4/25/11

L20: Markov processes and Random Walks

Outline:

- Review - Steady state for MP
- Reversibility for Markov processes
- Random walks
- Queueing delay in a G/G/1 queue
- Detection, decisions, & Hypothesis testing

If the embedded chain of a MP is positive recurrent, then

$$p_j = \frac{\pi_j/\nu_j}{\sum_k \pi_k/\nu_k}; \qquad \lim_{t\to\infty} \frac{M_i(t)}{t} = \frac{1}{\sum_k \pi_k/\nu_k} \quad \textbf{WP1}$$

where $M_i(t)$ is the sample-path average rate at which transitions occur WP1 and p_j is the sample-path average fraction of time in state j WP1, independent of starting state.

If $\sum_k \pi_k/\nu_k = \infty$, the transition rate $M_i(t)/t \to 0$ and the process has no meaningful steady state. Otherwise the steady state uniquely satisfies

$$p_j\nu_j = \sum_i p_i q_{ij}; \quad p_j > 0; \quad \textbf{all } j; \quad \sum_j p_j = 1$$

This says that rate in equals rate out for each state. For birth/death, $p_j q_{j,j+1} = p_{j+1} q_{j+1,j}$.

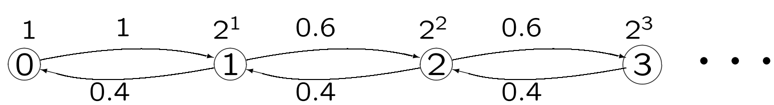
For an irreducible process, if there is a solution to the equations

$$p_j \nu_j = \sum_i p_i q_{ij}; \quad p_j > 0; \quad \textbf{all } j; \quad \sum_j p_j = 1$$

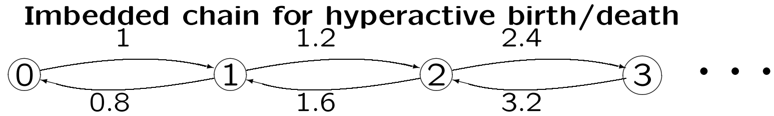
and if $\sum_i \nu_i p_i < \infty$, then the embedded chain is positive recurrent and

$$\pi_j = \frac{p_j \nu_j}{\sum_i p_i \nu_i}; \qquad \sum_i \pi_i / \nu_i = (\sum_i p_j \nu_j)^{-1}$$

If $\sum_i \nu_i p_i = \infty$, then each $\pi_j = 0$, the embedded chain is either transient or null-recurrent, and the notion of steady-state makes no sense.



Imbedded chain for hyperactive birth/death



Same process in terms of $\{q_{ij}\}$

Using $p_j q_{j,j+1} = p_{j+1} q_{j+1,j}$, we see that $p_{j+1} = \frac{3}{4} p_j$, so

$$p_j = (1/4)\,(3/4)^j \quad \textbf{and} \quad \sum_j p_j \nu_j = \infty.$$

If we truncate this process to k states, then

$$p_j \;=\; \frac{1}{4}\left(1 - \left(\frac{3}{4}\right)^k\right)\left(\frac{3}{4}\right)^j; \quad \pi_j = \frac{1}{3}\left(1 - \left(\frac{2}{3}\right)^k\right)\left(\frac{2}{3}\right)^{k-j}$$

$$\sum_j p_j \nu_j \;=\; \frac{1}{2}\left(1 - \left(\frac{3}{4}\right)^k\right)\left(\left(\frac{3}{2}\right)^k - 1\right) \;\to\; \infty$$

Reversibility for Markov processes

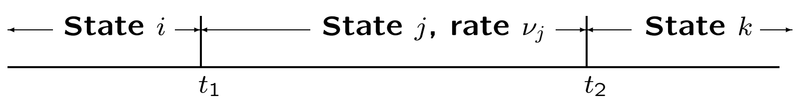
For any Markov chain in steady state, the backward transition probabilities P^*_{ij} are defined as

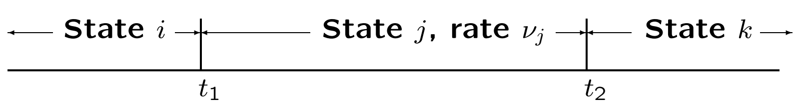
$$\pi_i P^*_{ij} = \pi_j P_{ji}$$

There is nothing mysterious here, just

$$\begin{aligned}\Pr\Big\{X_n = j, X_{n+1} = i\Big\} &= \Pr\Big\{X_{n+1} = i\Big\}\Pr\Big\{X_n{=}j|X_{n+1}{=}i\Big\} \\ &= \Pr\{X_n = j\}\Pr\Big\{X_{n+1}{=}i|X_n{=}j\Big\}\end{aligned}$$

This also holds for the embedded chain of a Markov process.





Moving right, after entering state j, the exit rate is ν_j, i.e., we exit in each δ with probability $\nu_j\delta$. The same holds moving left.

That is, a Poisson process is clearly reversible from the incremental definition.

Thus $\{\pi_i\}$ and $\{\nu_i\}$ are the same going left as going right

Note that the probability of having a (right) transition from state j to k in $(t, t+\delta)$ is $p_j q_{jk} \delta$. Similarly, if q^*_{kj} is the left-going process transition rate, the probability of having the same transition is $p_k q^*_{kj}$. Thus

$$p_j q_{jk} = p_k q^*_{kj}$$

By fiddling equations, $q^*_{kj} = \nu_k P^*_{kj}$.

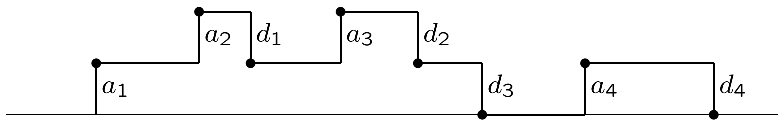
Def: A MP is reversible if $q^*_{ij} = q_{ij}$ for all i, j

Assuming positive recurrence and $\sum_i \pi_i/\nu_i < \infty$, the MP process is reversible if and only if the embedded chain is.

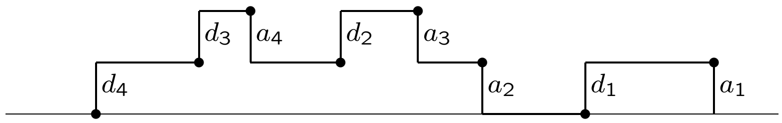
The guessing theorem: Suppose a MP is irreducible and $\{p_i\}$ is a set of probabilities that satisfies $p_i q_{ij} = p_j q_{ji}$ for all i, j and satisfies $\sum_i p_i \nu_i < \infty$.

Then (1), $p_i > 0$ for all i, (2), p_i is the sample-path fraction of time in state i WP1, (3), the process is reversible, and (4), the embedded chain is positive recurrent.

Useful application: All birth/death processes with $\sum_j p_j \nu_j < \infty$ are reversible. Similarly, if the Markov graph is a tree with $\sum_j p_j \nu_j < \infty$, the process is reversible.

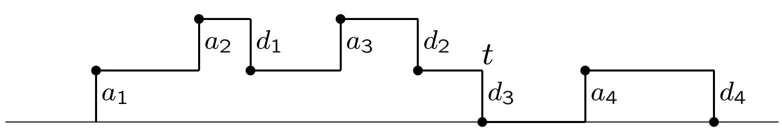


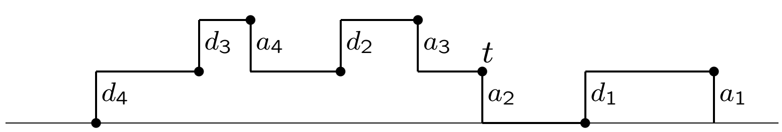
Right moving (forward) M/M/1 process



Left moving (backward) M/M/1 process

Burke's thm: Given an M/M/1 queue in steady-state with (arrival rate) $\lambda < \mu$ (departure rate),
(1) Departure process is Poisson with rate λ
(2) State $X(t)$ is independent of departures before t
(3) For FCFS, a customer's arrival time, given its departure at t, is independent of departures before t.

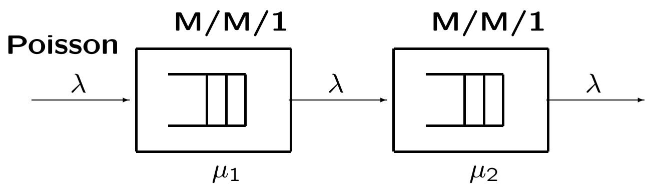




A departure at t in (right-moving) sample path is an arrival in the M/M/1 (left-moving) sample path.

For FCFS left-moving process, departure time of arrival at t depends on arrivals (and their service req.) to the right of t; independent of arrivals to left.

For corresponding (right-moving) process, the arrival time of that departure is independent of departures before t.



Consider tandem M/M/1 queues. Departures from first are Poisson with rate λ. Assume service times at rates μ_1 and μ_2, independent from queue to queue and independent of arrivals at each.

Arrivals at queue 2 are Poisson at rate λ by Burke and are independent of service times at 2. Thus the second queue is M/M/1.

The states of the two systems are independent and the time of a customer in system 1 is independent of that in 2.

Random walks

Def: Let $\{X_i;\, i \geq 1\}$ be a sequence of IID rv's, and let $S_n = X_1 + X_2 + \cdots + X_n$ for $n \geq 1$. The integer-time stochastic process $\{S_n;\, n \geq 1\}$ is called a random walk, or, specifically, the random walk based on $\{X_i;\, i \geq 1\}$.

We are used to sums of IID rv's, but here the interest is in the process. We ask such questions as:

1) Threshold crossing: For given $\alpha > 0$, what is the probability that $S_n \geq \alpha$ for at least one $n \geq 1$; what is the smallest n for which this crossing happens; and what is the overshoot $S_n - \alpha$?

2) Two thresholds: For given $\alpha > 0, \beta < 0$, what is the probability that $\{S_n;\, n \geq 1$ crosses α before it crosses β, and what is the n at which the first such crossing occurs?

These threshold-crossing problems are important in studying overflow in queues, errors in digital communication systems, hypothesis testing, ruin and other catastrophes, etc.

In many of the important applications, the relevant probabilities are very small and the problems are known as large deviation problems.

Moment generating functions and their use in upper bounds on these small probabilities are important here.

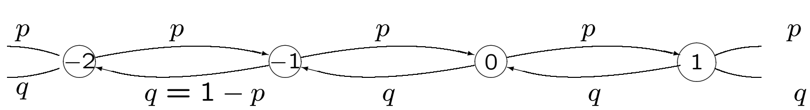
We start with a brief discussion of 3 simple cases: simple random walks, integer random walks, and renewal processes.

Simple random walks

A random walk (RW) $\{S_n; n \geq 1\},\ S_n = X_1 + \cdots + X_n$ is simple if X_n is binary with $\mathsf{p}_X(1) = 1,\ \mathsf{p}_X(-1) = q = 1-p$. This is just a scaling variation on a Bernoulli process. The probability that $X_i = 1$ for m out of n trials is

$$\Pr\{S_n = 2m - n\} = \frac{n!}{m!(n-m)!}\ p^m(1-p)^{n-m}.$$

Viewed as a Markov chain,

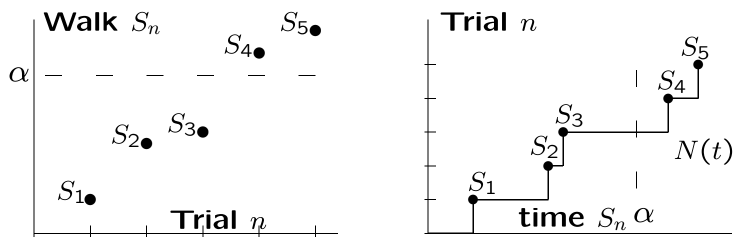


As in 'stop when you're ahead'

$$\Pr\left\{\bigcup_{n=1}^{\infty}\{S_n \geq k\}\right\} = \left(\frac{p}{1-p}\right)^k \qquad \textbf{if } p \leq 1/2.$$

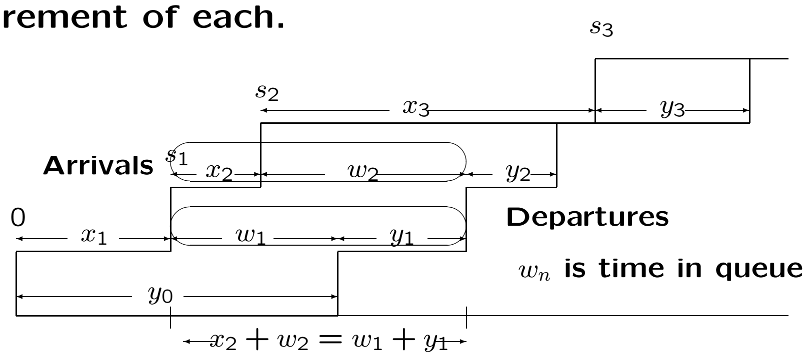
Integer RW's (where X is an integer rv) are similar. An integer RW can also be modeled as a Markov chain, but there might be an overshoot when crossing a treshold and the analysis is much harder.

Renewal processes are also special cases of random walks where X is a positive rv. When sketching sample paths, the axes are usually reversed from RP to RW.



Queueing delay in a G/G/1 queue

Let $\{X_i;\, i \geq 1\}$ be the (IID) interarrival intervals of a G/G/1 queue and let $\{Y_i;\, i \geq 1\}$ be the (IID) service requirement of each.



If arrival n is queued (e.g., arrival 2 above), then

$$x_n + w_n = y_{n-1} + w_{n-1}$$

If arrival n sees an empty queue, then $w_n = 0$.

$$w_n = y_{n-1} - x_n + w_{n-1} \ \textbf{if} \ w_{n-1} + y_{n-1} \ge x_n \ \textbf{else} \ w_n = 0$$

$$w_n = \max[w_{n-1} + y_{n-1} - x_n, \ 0]$$

Since this is true for all sample paths,

$$W_n = \max[W_{n-1} + Y_{n-1} - X_n, \ 0]$$

Define $U_n = Y_{n-1} - X_n$**. Then**

$$W_n = \max[W_{n-1} + U_n, \ 0]$$

Without the max, $\{W_n;\, n \ge 1\}$ **would be a random walk based on** $\{U_i; i \ge 1\}$**.**

With the max, $\{W_n;\, n \ge 1\}$ **is like a random walk, but it resets to 0 every time it goes negative. The text restates this in an alternative manner.**

Detection, decisions, & Hypothesis testing

These are different names for the same thing. Given observations, a decision must be made between a set of alternatives.

Here we consider only binary decisions, i.e., a choice between two hypotheses.

Consider a sample space containing a rv H **(the hypothesis) with 2 possible values,** $H = 0$ **and** $H = 1$**. The PMF for** H**,** $\mathsf{p}_H(0) = p_0$, $\mathsf{p}_H(1) = p_1$**, is called the a priori probabilities of** H**.**

Assume n **observations,** $Y_1, \dots, Y_n$ **are made. These are IID conditional on** $H = 0$ **and IID conditional on** $H = 1$**. Assume a pdf**

$$\mathsf{f}_{\vec{Y}|H}(\vec{y} \mid \ell) = \prod_{i=1}^{n} \mathsf{f}_{Y|H}(y_i \mid \ell).$$

By Baye's law,

$$\Pr\{H{=}\ell \mid \vec{y}\} = \frac{p_\ell f_{\vec{Y}|H}(\vec{y} \mid \ell)}{p_0 f_{\vec{Y}|H}(\vec{y} \mid 0) + p_1 f_{\vec{Y}|H}(\vec{y} \mid 1)}.$$

Comparing $\Pr\{H{=}0 \mid \vec{y}\}$ **and** $\Pr\{H{=}1 \mid \vec{y}\}$**,**

$$\frac{\Pr\{H{=}0 \mid \vec{y}\}}{\Pr\{H{=}1 \mid \vec{y}\}} = \frac{p_0 f_{\vec{Y}|H}(\vec{y} \mid 0)}{p_1 f_{\vec{Y}|H}(\vec{y} \mid 1)}.$$

The probability that $H = \ell$ **is the correct hypothesis, given the observation, is** $\Pr\left\{H{=}\ell \mid \vec{Y}\right\}$**. Thus we maximize the a posteriori probability of choosing correctly by choosing the maximum over** ℓ **of** $\Pr\left\{H{=}\ell \mid \vec{Y}\right\}$**. This is called the MAP rule (maximum a posteriori probability). It requires knowing** p_0 **and** p_1**.**