Review of Doppler Spread

The response to $\exp[2\pi ift]$ **is** $\hat{h}(f,t)\exp[2\pi ift]$**.**

$$\hat{h}(f,t) = \sum_j \beta_j \exp[-2\pi if\tau_j(t)] = \sum_j \exp[2\pi i\mathcal{D}_j t - 2\pi if\tau_j^o]$$

Define

$$\mathcal{D} = \max\mathcal{D}_j - \min\mathcal{D}_j; \qquad \mathcal{T}_{\mathsf{coh}} = \frac{1}{2\mathcal{D}}$$

The fading at f **is**

$$|\hat{h}(f,t)| = \left|\sum_j \exp[2\pi i(\mathcal{D}_j - \Delta)t - 2\pi if\tau_j^o]\right|.$$

Let $\Delta = (\max\mathcal{D}_j + \min\mathcal{D}_j)/2$**. The fading is the magnitude of a waveform baseband limited to** $\mathcal{D}/2$**.** $\mathcal{T}_{\mathsf{coh}}$ **is a gross estimate of the time over which the fading changes significantly.**

Review of time Spread

$$\hat{h}(f, t) = \sum_j \beta_j \exp[-2\pi i f \tau_j(t)]$$

For any given t, define

$$\mathcal{L} = \max \tau_j(t) - \min \tau_j(t); \qquad \mathcal{F}_{\text{coh}} = \frac{1}{2\mathcal{L}}$$

The fading at f is

$$|\hat{h}(f, t)| = \left| \sum_j \exp[2\pi i (\tau_j(t) - \tau')f] \right| \qquad (\textbf{ind. of } \tau')$$

Let $\tau' = \tau_{\text{mid}} = (\max \tau_j(t) + \min \tau_j(t))/2$. The fading is the magnitude of a function of f with transform limited to $\mathcal{L}/2$. $\mathcal{T}_{\text{coh}}$ is a gross estimate of the frequency over which the fading changes significantly.

Baseband system functions

The baseband response to a complex baseband input $u(t)$ is

$$v(t) \;=\; \int_{-W/2}^{W/2} \widehat{u}(f)\widehat{h}(f+f_c,t)\, e^{2\pi i(f-\triangle)t}\, df$$

$$= \int_{-W/2}^{W/2} \widehat{u}(f)\, \widehat{g}(f,t)\, e^{2\pi ift}\, df$$

where $\widehat{g}(f,t) = \widehat{h}(f+f_c,t)e^{-2\pi i\triangle t}$ is the baseband system function and $\triangle = \tilde{f}_c - f_c$ is the frequency offset in demodulation.

By the same relationship between frequency and time we used for bandpass,

$$v(t) = \int_{-\infty}^{\infty} u(t-\tau)g(\tau,t)\, d\tau$$

$$\widehat{h}(f,t) \;=\; \sum_j \beta_j \exp\{-2\pi i f \tau_j(t)\}$$

$$\widehat{g}(f,t) \;=\; \sum_j \beta_j \exp\{-2\pi i (f+f_c)\tau_j(t) - 2\pi i \Delta t\}$$

$$\widehat{g}(f,t) \;=\; \sum_j \gamma_j(t) \exp\{-2\pi i f \tau_j(t)\} \qquad \textbf{where}$$

$$\gamma_j(t) \;=\; \beta_j \exp\{-2\pi i f_c \tau_j(t) - 2\pi i \Delta t\}$$

$$\;=\; \beta_j \exp\{2\pi i [\mathcal{D}_j - \Delta]t - 2\pi i f_c \tau_j^o$$

$$g(\tau,t) \;=\; \sum_j \gamma_j(t)\delta(\tau - \tau_j(t))$$

$$v(t) \;=\; \sum_j \gamma_j(t)u(t - \tau_j(t))$$

Flat fading

Flat fading is defined as fading where the band-width $W/2$ of $u(t)$ is much smaller than $\mathcal{F}_{\text{coh}}$.

For $|f| < W/2 << \mathcal{F}_{\text{coh}}$,

$$\widehat{g}(f,t) = \sum_j \gamma_j(t) \exp\{-2\pi i f \tau_j(t)\} \approx \widehat{g}(0,t) = \sum_j \gamma_j(t)$$

$$v(t) = \int_{-W/2}^{W/2} \widehat{u}(f)\,\widehat{g}(f,t)\,e^{2\pi i f t}\,df \approx u(t) \sum_j \gamma_j(t)$$

Equivalently, $u(t)$ is approximately constant over intervals much less than $\mathcal{L}$.

$$v(t) = \sum_j \gamma_j(t) u(t - \tau_j(t)) = u(t) \sum_j \gamma_j(t)$$

Discrete-time baseband model $(T = 1/W)$

$$u(t) = \sum_n u(nT)\mathsf{sinc}(t/T - n)$$

$$v(t) = \int u(t - \tau)g(\tau, t)\, d\tau$$

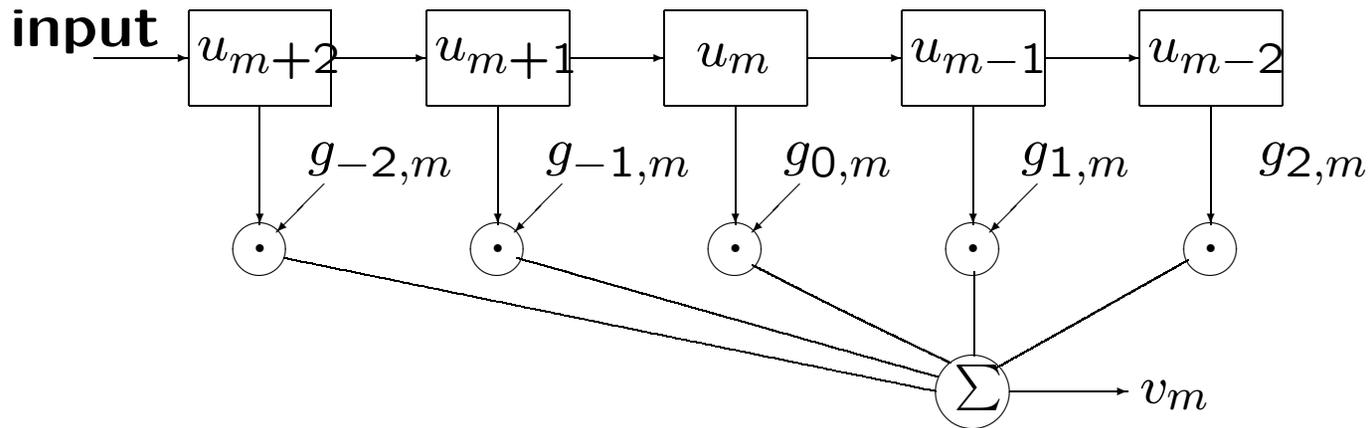$$= \sum_n u(nT) \int g(\tau, t)\,\mathsf{sinc}(t/T - \tau/T - n)\, d\tau$$

$$v(mT) = \sum_k u(mT - kT) \int g(\tau, mT)\,\mathsf{sinc}(k - \tau/T)\, d\tau$$

Letting $u_n = u(nT)$ **and** $v_n = v(nT)$,

$$v_m = \sum_k g_{k,m}\, u_{m-k}; \qquad g_{k,m} = \int g(\tau, mT)\,\mathsf{sinc}(k - \tau/T)\, d\tau.$$

Since $g(\tau, t) = \sum_j \gamma_j(t)\delta(\tau - \tau_j(t))$,

$$g_{k,m} = \sum_j \gamma_j(mT)\,\mathsf{sinc}\left[k - \frac{\tau_j(mT)}{T}\right]$$

$$g_{k,m} = \sum_{j} \gamma_j(mT) \, \mathsf{sinc}\left[k - \frac{\tau_j(mT)}{T}\right]$$

Each $g_{k,m}$ gets contributions from each path, but the major contributions are from paths with $\tau_j(mT) \approx kT$.

For flat fading, only one tap, $g_{0,m}$, is significant.

Stochastic channel model

For many paths and a small number of taps $(T \sim \mathcal{L})$, many paths contribute to each channel filter tap.

View $g_{k,m}$ as a sum of many unrelated paths.

View $g_{k,m}$ as a sample value of a rv $G_{k,m}$ with zero mean iid Gaussian real and imaginary parts, G_r, G_i

$$f_{G_r,G_i}(g_r, g_i) = \frac{1}{2\pi\sigma_k^2} \exp\left\{ \frac{-g_r^2 - g_i^2}{2\sigma_k^2} \right\}$$

$G_{k,m}$ has independent magnitude and phase.

Phase is uniform; magnitude has Rayleigh den–
sity

$$f_{|G_{k,m}|}(|g|) = \frac{|g|}{\sigma_k^2} \exp\left\{\frac{|g|^2}{2\sigma_k^2}\right\}$$

This is quite a flaky modeling assumption since
there are often not many paths.

If we look at the ensemble of all uses of cellular
systems (or some other kind of system), the
model makes much more sense.

Basically, this is just a simple model to help
understand a complex situation.

Another common model is the Rician distribution. There is one path with large known magnitude but random phase, plus other complex Gaussian paths.

The phase is uniformly distributed and independent of the amplitude, which is quite messy.

If the large known path has both amplitude and phase known, the resulting amplitude distribution of fixed plus Gaussian terms is still Rician.

The Rician model has the same problems as the Rayleigh model.

Tap gain correlation function

How does $G_{k,m}$ varies with k and m?

Assume independence for $k \neq k'$.

These quantities refer to well separated paths.

A high velocity path at range k could move to k', but we ignore this.

Simplest statistical measure is tap gain correlation,

$$R(k,n) = \mathsf{E}\Big[G_{k,m}G^*_{k,m+n}\Big]$$

This is assumed to not depend on m (WSS). With joint Gaussian assumption on taps, have stationarity.
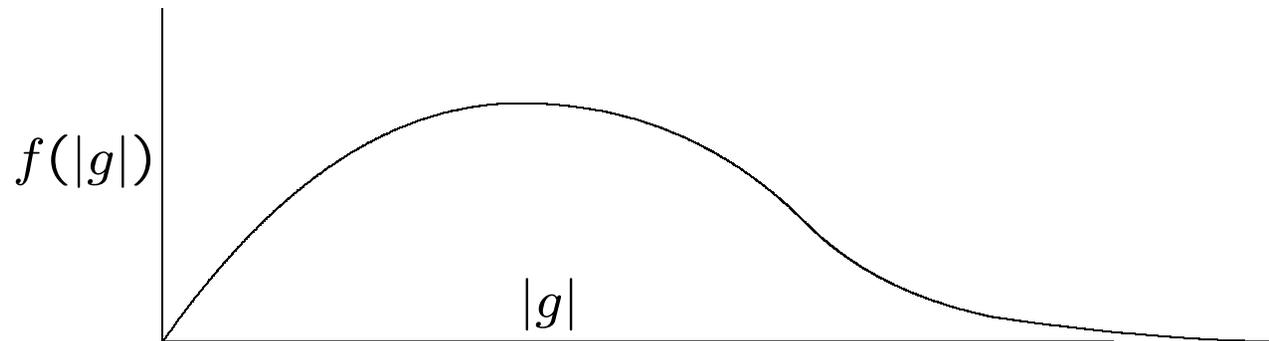
Flat Rayleigh fading

Assume a single tap model with $G_{0,m} = G_m$. Assume G_m is circ. symmetric Gaussian with $\mathbf{E}[|G_m|^2] = 1$.

The magnitude is Rayleigh with

$$f_{|G_m|}(|g|) = 2|g|\exp\{-|g|^2\} \qquad ; \qquad |g| \geq 0$$

$$V_m = U_m G_m + Z_m; \qquad \Re(Z_m), \Im(Z_m) \sim \mathcal{N}(0, W N_0/2)$$

Antipodal binary communication does not work here. It can be viewed as phase modulation (180°) and the phase of V_m is independent of U_m.

We could use binary modulation with $U_m = 0$ or a, but it is awkward and unsymmetric.

Consider pulse position modulation over 2 samples.

$$H = 0 \quad \longrightarrow \quad (U_0, U_1) = (a, 0)$$
$$H = 1 \quad \longrightarrow \quad (U_0, U_1) = (0, a).$$

This is equivalent to any binary scheme which uses 2 symmetric complex degrees of freedom, modulating by choice among degrees.

$$H = 0 \quad \longrightarrow \quad V_0 = aG_0 + Z_0; \quad V_1 = Z_1$$

$$H = 1 \quad \longrightarrow \quad V_0 = Z_0; \quad V_1 = aG_1 + Z_1.$$

$$H = 0 \quad \longrightarrow \quad V_0 \sim \mathcal{N}_c(0, \, a^2 + WN_0); \quad V_1 \sim \mathcal{N}_c(0, \, WN_0)$$

$$H = 1 \quad \longrightarrow \quad V_0 \sim \mathcal{N}_c(0, \, WN_0); \quad V_1 \sim \mathcal{N}_c(0, \, a^2 + WN_0).$$

$\mathcal{N}_c(0, \sigma^2)$ means iid real, imaginary, each $\mathcal{N}(0, \sigma^2/2)$.

$$H = 0 \quad \longrightarrow \quad V_0 \sim \mathcal{N}_c(0,\ a^2 + WN_0); \quad V_1 \sim \mathcal{N}_c(0,\ WN_0)$$
$$H = 1 \quad \longrightarrow \quad V_0 \sim \mathcal{N}_c(0,\ WN_0); \quad V_1 \sim \mathcal{N}_c(0,\ a^2 + WN_0).$$

$$f(v_0, v_1 | H{=}0) \;=\; \alpha \exp\left\{ -\frac{|v_0|^2}{a^2 + WN_0} - \frac{|v_1|^2}{WN_0} \right\}$$

$$f(v_0, v_1 | H{=}1) \;=\; \alpha \exp\left\{ -\frac{|v_0|^2}{WN_0} - \frac{|v_1|^2}{a^2 + WN_0} \right\}$$

$$LLR(v_0, v_1) \;=\; \ln\left\{ \frac{f(v_0, v_1 | H_0)}{f(v_0, v_1 | H_1)} \right\} \;=\; \frac{\left[|v_0|^2 - |v_1|^2 \right] a^2}{(a^2 + WN_0)(WN_0)}$$

Given $H = 0$, $|V_0|^2$ is exponential, mean $a^2 + WN_0$ and $|V_1|^2$ is exponential, mean WN_0. Error if the sample value for the first less than that of the second.

Let $X_0 = |V_0|^2$ and $X_1 = |V_1|^2$. Given H_0, X_1 is an exponential rv with mean WN_0 and X_0 is exponential with mean $WN_0 + a^2$.

Error if $X_0 < X_1$.

Let $\tilde{X} = X_1 - X_0$. For $\tilde{X}_1 > 0$,

$$f(\tilde{x}|H_0) = \int_{x_1 = \tilde{x}}^{\infty} f_1[x_1|H_0] \, f_0[(x_1 - \tilde{x})|H_0] \, dx_1$$

$$= \frac{1}{a^2 + 2WN_0} \exp\left\{-\frac{\tilde{x}}{WN_0}\right\}.$$

$$\Pr(e|H_0) = \frac{WN_0}{a^2 + 2WN_0} = \frac{1}{2 + a^2/(WN_0)}$$

$$= \frac{1}{2 + E_b N_0}$$

We next look at non-coherent detection. We use the same model except to assume that $|g_0| = |g_1| = \tilde{g}$ is known. We calculate $\Pr(e)$ conditional on $\tilde{g}$.

We find that knowing $\tilde{g}$ does not aid detection. We also see that the Rayleigh fading result occurs because of the fades rather than lack of knowledge about them.

$$H = 0 \quad \longrightarrow \quad V_0 = a\tilde{g}e^{i\phi_0} + Z_0; \quad V_1 = Z_1$$
$$H = 1 \quad \longrightarrow \quad V_0 = Z_0; \quad V_1 = a\tilde{g}e^{i\phi_1} + Z_1.$$

The phases are independent of H, so $|V_0|$ and $|V_1|$ are sufficient statistics.

The ML decision is $\hat{H} = 0$ if $|V_0| \geq |V_1|$. This decision does not depend on $\tilde{g}$.

Since the phase of $\tilde{g}$ and that of the noise are independent, we can choose rectangular coordinates with real $\tilde{g}$. The calculation is straightforward but lengthy.

$$\Pr(e) = \frac{1}{2}\exp\left(\frac{-a^2\tilde{g}^2}{2WN_0}\right) = \frac{1}{2}\exp\left(\frac{-E_b}{2N_0}\right)$$

If the phase is known at the detector,

$$\Pr(e) = Q\left(\frac{a^2\tilde{g}^2}{WN_0}\right) \leq \sqrt{\frac{N_0}{2\pi E_b}}\exp\left(\frac{-E_b}{2N_0}\right)$$

When the exponent is large, the db difference in E_b to get equality is small.

CHANNEL MEASUREMENT

Channel measurement <u>is</u> <u>not</u> very useful at the receiver for single bit transmission in flat Rayleigh fading.

It <u>is</u> useful for modifying transmitter rate and power.

It <u>is</u> useful when diversity is available.

It <u>is</u> useful if a multitap model for channel is appropriate. This provides a type of diversity (each tap fades approximately independently).

Diversity results differ greatly depending on whether receiver knows channel and transmitter knows channel.

SIMPLE PROBING SIGNALS

Assume k_0 **channel taps,** $G_{0,m}, \dots, G_{k_0-1,m}$.
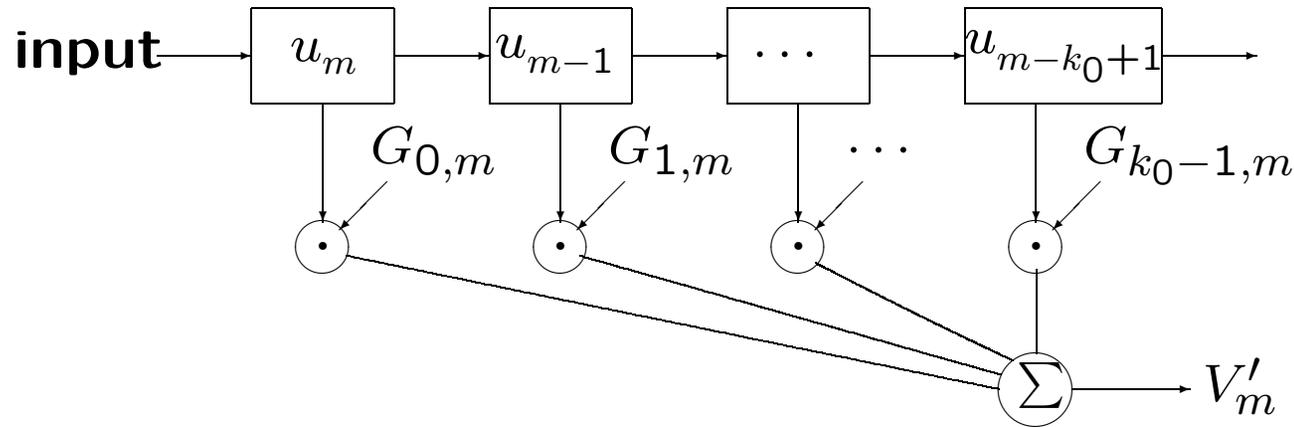
$$V'_m = u_m G_{0,m} + u_{m-1} G_{1,m} + \cdots + u_{m-k_0+1} G_{k_0-1,m}$$
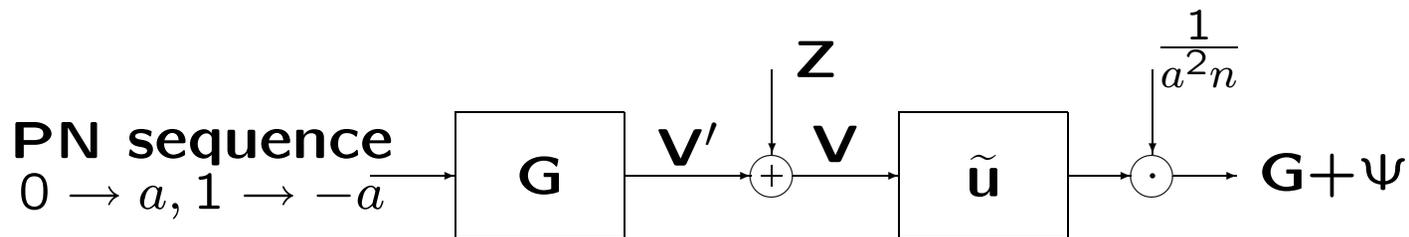
Send $(a, 0, 0, \dots, 0)$

$$\mathbf{V}' = (aG_{0,0}, aG_{1,1}, \dots, aG_{k_0-1,k_0-1}, 0, 0, \dots, 0)$$

$V_m = V'_m + Z_m$. **Estimate** $G_{m,m}$ **as** V_m/a. **Estimation error is** $\mathcal{N}_c(0, WN_0/a^2)$.

Pseudonoise (PN) PROBING SIGNALS

A PN sequence is a binary sequence that appears to be iid. It is generated by a binary shift register with the mod-2 sum of given taps fed back to the input. With length k, it generates all $2^k - 1$ binary non-zero k-tuples and is periodic with length $2^k - 1$.
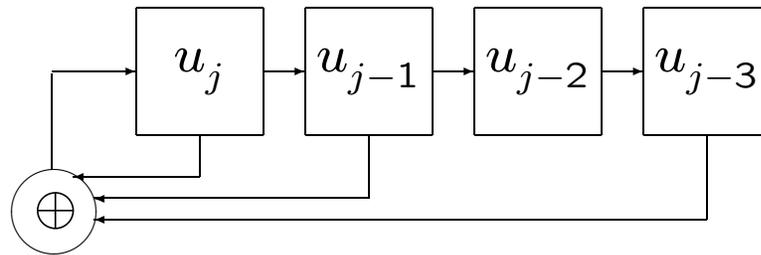
PN sequence
$0 \to a,\, 1 \to -a$ $\longrightarrow$ $\boxed{\textbf{G}}$ $\xrightarrow{\textbf{V}'}$ $\oplus \xleftarrow{\textbf{Z}}$ $\xrightarrow{\textbf{V}}$ $\boxed{\tilde{\textbf{u}}}$ $\longrightarrow$ $\odot \xleftarrow{\frac{1}{a^2 n}}$ $\longrightarrow$ **G+Ψ**

u is $\approx$ orthogonal to each shift of u so

$$\sum_{m=1}^{n} u_m u_{m+k}^* \approx \begin{cases} a^2 n & ; \quad k = 0 \\ 0 & ; \quad k \neq 0 \end{cases} = a^2 n \delta_k$$

If $\tilde{\textbf{u}}$ is matched filter to u, then $\textbf{u} * \tilde{\textbf{u}} = a^2 n \delta_j$.
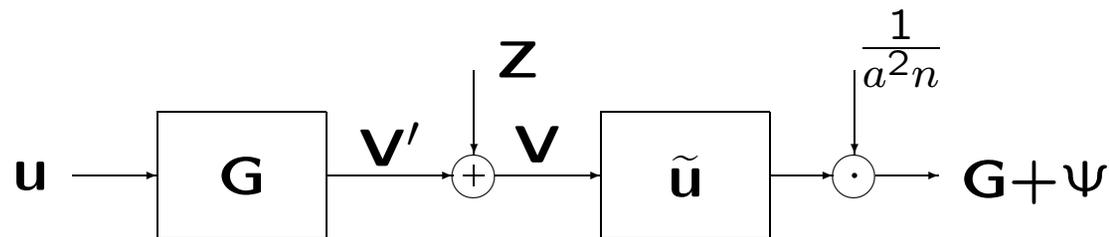
Binary feedback shift register

Periodic with period $15 = 2^4 - 1$

If $\mathbf{u} * \tilde{\mathbf{u}} = a^2 n \delta_j$, then

$$\mathbf{V}' * \tilde{\mathbf{u}} = (\mathbf{u} * \mathbf{G}) * \tilde{\mathbf{u}} = (\mathbf{u} * \tilde{\mathbf{u}} * \mathbf{G}) = a^2 n \mathbf{G}$$

The PN property has the same effect as using a single input surrounded by zeros.

The response at time m of $\tilde{u}$ to $\mathbf{Z}$ is the sum of n iid rv's each of variance $a^2 N_0 W$.

The sum has variance $a^2 n N_0 W$. After scaling by $1/(a^2 n)$, $\mathsf{E}[|\Psi_k|^2] = \frac{N_0 W}{a^2 n}$.

The output is a ML estimate of G; MSE decreases with n
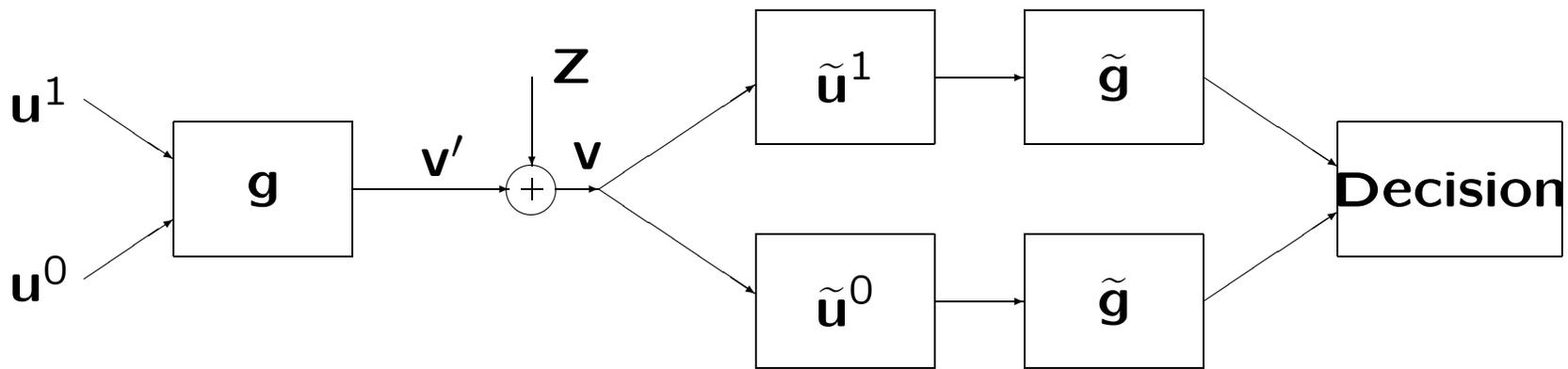
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RAKE RECEIVER

The idea here is to measure the channel and make decisions at the same time.

Assume a binary input, $H=0 \rightarrow \mathbf{u}^0$ and $H=1 \rightarrow \mathbf{u}^1$

With a known channel g, the ML decision is based on pre-noise inputs $\mathbf{u}^0 * \mathbf{g}$ and $\mathbf{u}^1 * \mathbf{g}$.

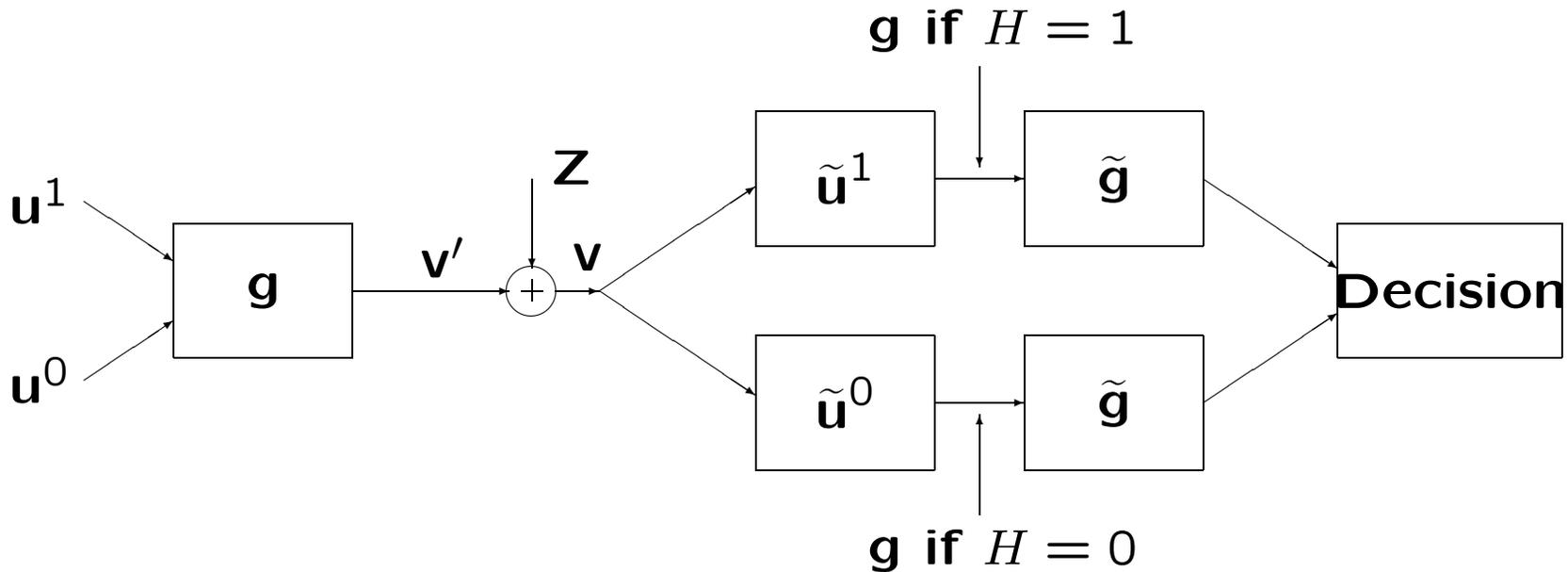$$\Re(\langle \mathbf{v}, \mathbf{u}^0 * \mathbf{g} \rangle) \overset{\widehat{H}=0}{\underset{\widehat{H}=1}{\gtrless}} \Re(\langle \mathbf{v}, \mathbf{u}^1 * \mathbf{g} \rangle).$$

We can detect using filters matched to $\mathbf{u}^0 * \mathbf{g}$ and $\mathbf{u}^1 * \mathbf{g}$

Note the similarity of this to the block diagram for measuring the channel.

If the inputs are PN sequences (which are often used for spread spectrum systems), then if the correct decision can be made, the output of the corresponding arm contains a measurement of g.

u^1 and u^0 are non-zero from time 1 to n. v' is non-zero from 1 to $n+k_0-1$.

$\tilde{u}^1$ and $\tilde{u}^0$ are non-zero from $-n$ to -1 (receiver time).

If $H = 1$ or $H = 0$, then g plus noise appears from time 0 to $k_0 - 1$ where shown. Decision is made at time 0, receiver time.
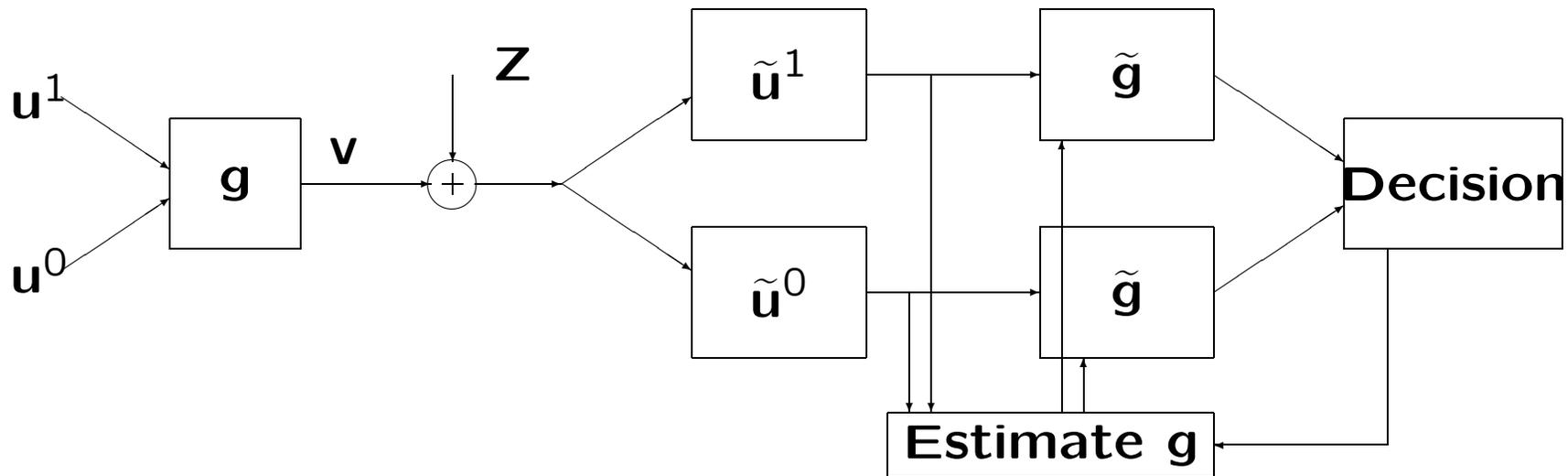
If $\hat{H} = 0$, then a noisy version of g probably exists at the output of the matched filter u^0. That estimate of g is used to update the matched filters $\widetilde{g}$.

If $\mathcal{T}_c$ is large enough, the decision updates can provide good estimates.

Suppose there is only one Rayleigh fading tap in the discrete-time model.

Suppose the estimation works perfectly and g is always known. Then the probability of error is the coherent error probability $Q(\sqrt{E_b/N_0})$ for orthogonal signals and $E_b = a^2 n |g|^2 / W$.
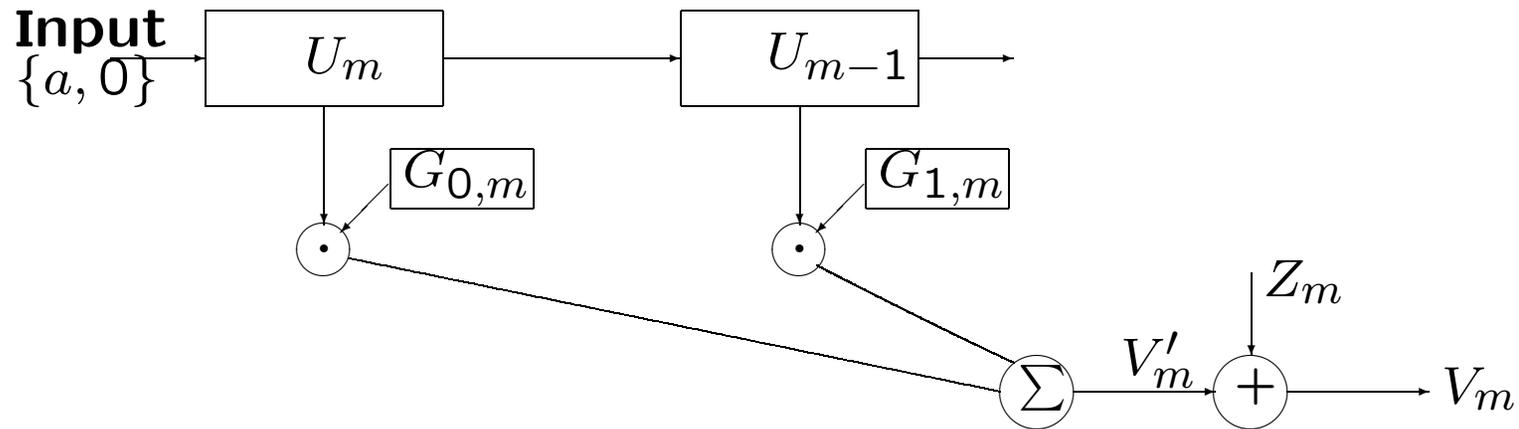
This is smaller than incoherent $\Pr(e) = \frac{1}{2}\exp\{-E_b/(2N_0)\}$.

Averaging over G, incoherent result is $\dfrac{1}{2+\bar{E}_b/N_0}$ and coherent result is at most half of this.

Measurement doesn't help here.

Diversity

Consider a two tap model. More generally con-
sider independent observations of the input.

Consider the input $H{=}0 \to a, 0, 0, 0$ and $H{=}1 \to 0, 0, a, 0$.

For $H{=}0$, $\mathbf{V}' = aG_{0,0}, aG_{1,1}, 0, 0$. For $H{=}1$, $\mathbf{V}' = 0, 0, aG_{0,2}, aG_{1,3}$.

Assume each G is $\mathcal{N}_c(0,1)$ and each Z is $\mathcal{N}_c(0,\sigma^2)$. Given $H=0$, V_1 and V_2 are $\mathcal{N}_c(0, a^2 + \sigma^2)$ and V_2, V_3 are $\mathcal{N}_c(0,\sigma^2)$.

Given $H=1$, V_1 and V_2 are $\mathcal{N}_c(0,\sigma^2)$ and V_2, V_3 are $\mathcal{N}_c(0, a^2+\sigma^2)$.

Sufficient statistic is $|V_j|^2$ for $1 \le j \le 4$. Even simpler, $|V_1|^2 + |V_2|^2 - |V_3|^2 - |V_4|^2$ is a sufficient statistic.

$$\Pr(e) = \frac{4 + 3\frac{a^2}{\sigma^2}}{\left(2 + \frac{a^2}{\sigma^2}\right)^3} = \frac{4 + \frac{3E_b}{2N_0}}{\left(2 + \frac{E_b}{2N_0}\right)^3}$$

This goes down with $(E_b/N_0)^{-2}$ as $(E_b/N_0) \to \infty$.

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